Chapter 3

Modeling Random Motion

WHAT DOES "RANDOM" MEAN? Think carefully before you answer! The definition may not be as obvious as you think.

> Q3.1: After checking the dictionary definition, consider the following four statements:
>
> The result of a coin flip is *not* random, because there are only two possible outcomes. *True or False?*
>
> The result of rolling a 6-sided die is *not* random, because there are only six possible outcomes. *True or False?*
>
> Whether I win the state Lottery or not *is* random, because there are so many people playing the Lottery at the same time. *True or False?*
>
> The weather *is* random, because so many conditions affect the weather that we cannot predict it. *True or False?*

The main theme of this book is the study of how order grows out of randomness. Every structure in your body grows and every process in your body takes place in the presence of randomly-agitated molecules. Yet instead of being torn apart by this randomness, we survive. We even *thrive* on the randomness of nature. How can this be? Before we can begin to answer this question, we must study randomness itself, and details of the staggering, zigzag paths that atoms and molecules execute all around us.

Can order grow out of randomness? Think about the following question:

> Q3.2: Consider a group of 20 people. We want to divide the group into two groups, group A and group B. Each group should have 10 members. Now flip a coin for each person: heads, the person goes to group A; tails, the person goes to group B. Will the people end up evenly divided, ten in each group? Could they all end up in one group? Which of these results is more likely?

3.1 Measuring Randomness?

Is the present always influenced by the past? Suppose you are flipping a coin and, by chance, flip three heads in a row. Does flipping three heads in a row affect the next flip—the fourth flip—or not? Is the fourth flip *more* likely to be another head? Or is the fourth flip *less* likely to be a head?

- Do you believe in "winning streaks," meaning that three heads in a row is more likely to lead to a *head* on the next flip? *Why* do you believe in winning streaks?

- Or do you expect your "luck to run out," meaning that three heads in a row is more likely to be followed by a *tail* on the next flip? *Why* do you believe that your luck can run out?

- Or do you expect equal chances of getting a head or a tail on the next flip, independent of what happened before? *Why* do you believe that the next flip is independent?

HandsOn 9: Lottery Game

Flip a coin over and over again until you get three heads in a row. Now choose one of these strategies and stick to it.

- Strategy #1: If you believe in a winning streak, bet that the next flip will be a head.
- Strategy #2: If you believe in luck running out, bet that the next flip will be a tail.
- Strategy #3: If you believe the next flip is random, bet on heads the first time, bet on tails the next, and so forth.

Now flip the coin a fourth time. How did it come out? Did you win or lose?

Again, flip the coin until you get three heads in a row, then make another bet, using the same strategy. Assume that a win brings you $1.00 and a loss costs you $1.00. Carry out this procedure again and again. Keep track of how much "money" you win or lose.

> Q3.3: Can we speed up this process? Suppose you flip three different coins at once by shaking them in your cupped hands and throwing them on the table. Then just look to see if all three are heads. If not, shake them up and throw them down again and again until all three do come up heads. Then flip a fourth coin by hand and see if you win or lose according to your chosen strategy. Discuss the following question: Do you expect that this will give the same result as flipping a fourth time after the *same* coin has come up heads three times in a row?

Winning Streak is also available as a Java applet.

Now move on to the computer program called **Winning Streak?** This time the computer program allows you to bet on the outcome of a coin-flip after four coins in a row land on the same side (four heads or four tails). With the computer flipping coins, you can test your strategy faster than you could flipping coins by hand. What is the result?

Is your strategy a winner? or a loser? Or do you break even? If your friends are doing the same activity, pool your results in order to compare the success rates of different strategies. Continue your discussion until there is general agreement that (i) winning streaks exist, (ii) losing streaks exist, or (iii) the next flip cannot be predicted: each flip is random.

END ACTIVITY

3.2 Observed Distributions

No one can say with certainty what will happen next in a random process such as a coin flip. Even so, some kinds of predictions are possible and useful. If you flip 10 coins, about how many of them do you expect to come up heads? Is it *possible* that all 10 will come up heads? Is 10 heads in a row *likely*? Is it *possible* that all 10 will come up tails? Is 10 tails in a row *likely*? To start answering these questions, carry out the following activity.

Q3.4: Suppose one thousand students are each given one penny. Each student flips his or her penny ten times and records the number of heads. How many out of 1000 students will flip 10 heads, how many will flip 9 heads, how many flip 8 heads, and so on? *Guess* the answers to these questions and fill in a copy of Table 3.1. Be prepared to describe the reasoning behind your guesses.

HandsOn 10: Coin Flipping

Now let's find out what is the real outcome of flipping a coin ten times. If you are using this book in a classroom setting, work in groups of two or three. Save time by "flipping" ten coins at once: Shake up ten coins between your cupped hands and throw them on the table. Count the number of heads and report this number to the group member who is keeping record. Shake up the ten coins and drop them again. Again

Table 3.1: *Guess* the results of 10 flips by each of 1000 students. Sketch your prediction by drawing a bar graph or histogram on a copy of Figure 3.1

No. of Heads	No. of Students (out of 1000)
10	
9	
8	
7	
6	
5	
4	
3	
2	
1	
0	

count the number of heads and report this number. Each group does this ten times.

> Q3.5: Does throwing down ten different coins give the same results as flipping a single coin ten times? What *assumptions* do you make in answering this question: assumptions about whether different coins are identical or not and assumptions about the independence of each coin flip?

Now combine the results for all the groups in the class. Have someone make a graph like the one in Figure 3.2 on the board. Place a large X on the graph for the result of each trial, stacking the X's on top of one another.

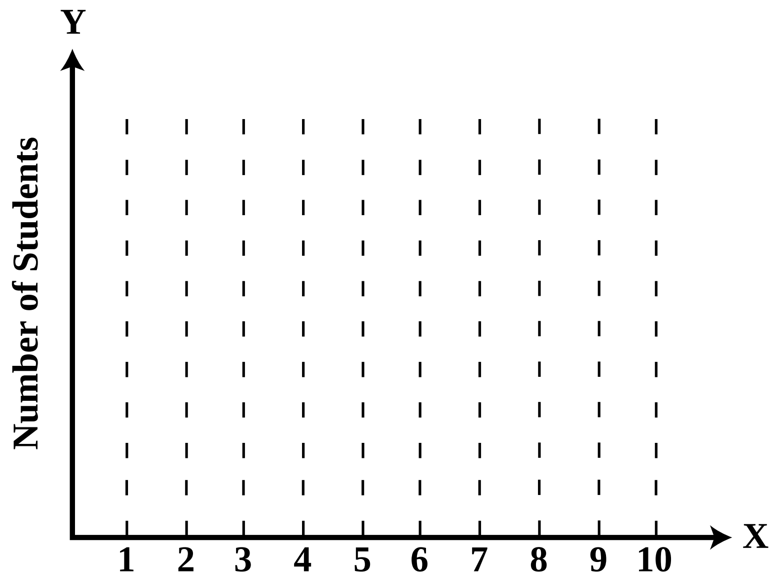

Figure 3.1: Bar graph (a simple histogram) of number of students vs. number of heads to be completed when the "process" of flipping coins is completed. Make your own scale (divisions) on the vertical axis.

Q3.6: Meet with your group and discuss the results plotted on the combined graph. Do they look like what you predicted? Is the graph uneven in shape? Why or why not? What would the graph look like after 1000 trials?

Q3.7: Did certain numbers of heads occur more often than others?

Q3.8: What number of heads is the *most likely* to occur? What fraction of the time does that actually happen?

Q3.9: Were there any trials that resulted in zero heads or ten heads?

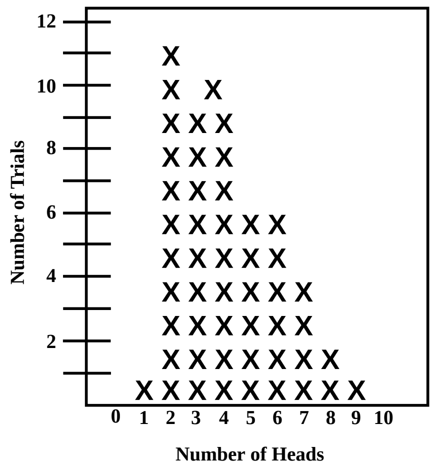

Figure 3.2: One possible histogram of the number of heads when 10 coins are flipped in 50 trials. Each X represents the result of one trial.

Q3.10: Why is the distribution lopsided? If we did this again, could the new distribution be lopsided the other way? Why or why not?

Q3.11: Based on this activity, would you predict the same histogram for 1000 trials as you guessed earlier? If not, what would you predict the histogram will look like for 1000 trials? Make new entries to the right of your copy of Table 3.1. In general, what do you expect will happen as the number of trials is increased? Would the distribution remain lopsided? Make a guess before going on to the computer activity where you will able to verify your prediction.

END ACTIVITY

SimuLab 3: Wholesale Coin Flipping

Random Walk is a program that uses a computer to do our ten-penny flip thousands of times faster than we can. The computer is programmed so that "heads" and "tails" are equally likely to occur. The computer also plots the results.

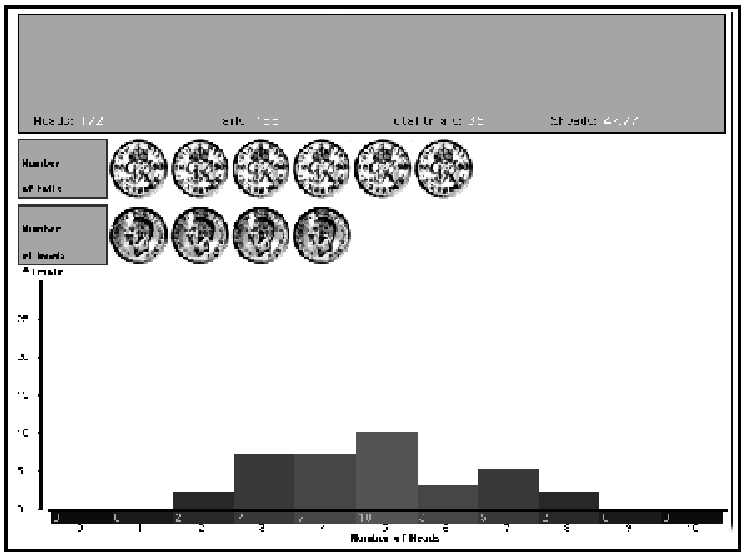

Figure 3.3: Bar graph (histogram) showing number of heads when 10 coins are flipped repeatedly.

1. Start the **Random Walk** program, which begins with the coin-flipping game (Figure 3.3)

2. Click on the **Flip** button to flip one coin at a time. Click **Flip** again. And again. Repeat until you have flipped ten coins. Study the numbers at the bottom and the top of the graph: What is being recorded?

3. Now repeat the process, but this time click **Go** to have the computer flip coins automatically, one after the other. Notice that as the bar graph (histogram) grows, the vertical scale changes to

keep the plot on the screen. The computer will run 100 trials of ten coins each. Record results in your notebook.

4. Now choose a new window (choose **Coin Flip** in the **New** box on the control panel).

5. Click **Go** and watch the graph grow. To speed up the process, choose **Less Graphics** from the **Options** menu.

6. Choose the **Tile Windows** command under the **Options** menu and compare the two graphs. (**Tile Windows** shrinks the windows and places them side by side, so you can view more than one at a time.) Are the two patterns the same? Can you predict the shape of the next graph if you make a third run? Try it! Tile the resulting three displays.

7. Click on **Coin Flip** one more time, select 500 for the **Number of Trials** under the **Experiment** menu, and choose **Less Graphics** under the **Options** menu. Press **Go**.

8. Select the **Tile Windows** command under **Options** to see all of these results side by side on the screen.

9. Experiment on your own. Change the number of trials. You can also change the number of flips per trial with the **Number of Steps** command under the **Experiment** menu. You should now be able to verify the prediction you made at the end of HandsOn 10. Again, make a prediction about what the graph will look like.

> Q3.12: As the number of trials increases how does the shape of the graph change? Is the *width* of the distribution greater after a larger number of trials? Is the bar graph *smoother* after 500 trials than after 100 trials?

> Q3.13: Does the number of trials or the number of flips affect the shape of the graph? How?

END ACTIVITY

3.3 Random Walks

In the preceding section we found that some predictability grows out of random coin flipping, leading to a "bell-shaped" bar graph of the results. Such a distribution is also called the *normal* or *Gaussian* distribution (you will encounter this distribution at many places in this book). This section carries the idea further, relating random coin flipping to random motion. Random movement is important for understanding the microscopic world in nature, because atoms and molecules move randomly. How can we describe the random motion of molecules in, say, a gas? Molecules are too small to see, so to help us think concretely we replace a molecule with something we can see: a wandering ant. If a wandering ant starts at a lamp post and takes steps of equal length along the street, how far will it be from the lamp post after a certain number, say N, steps? Though this question is seemingly trivial, it poses one of the most basic problems in statistical science.[1]

HandsOn 11: Ten-Step Random Walk

It is easiest to visualize random motion (random walk) along one line, that is, in one dimension. Call x the position of the ant (i.e., walker) on a one-dimensional line. Locate the origin, that is $x = 0$, at the lamp post. Then let each "step" of the ant—right or left along the line—be of equal length. One way to picture this is to use one row of a checkerboard or an enlarged photocopy of Figure 3.4.

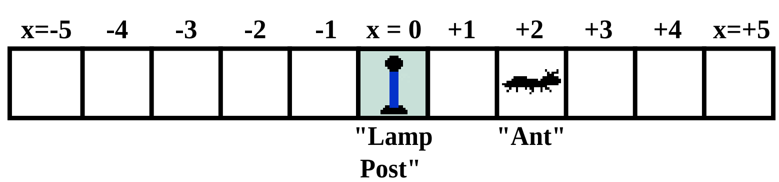

Figure 3.4: Diagram of a wandering ant. How many steps do you think he has already taken if his starting-point was at the "Lamp post"?

[1]W. Weaver, "Probability," *Scientific American* (October 1950); M. Kac, "Probability," *Scientific American* (September 1964); B. H. Lavenda, "Brownian Motion," *Scientific American* (February 1985).

Choose the *direction* of the step the ant will take by flipping a penny:

- If it is a head, the ant steps right and x increases by one.
- If it is a tail, the ant steps left and x decreases by one.

A head or tail is equally likely; therefore it is equally probable that the ant steps right or left.

Do this activity with a partner. Use a silver-colored coin (nickel, dime, or quarter) to represent the position of the ant. To begin, put the "ant" in a center cell (the position of the lamp post). The ant steps from one cell to the next, right or left randomly, depending on whether the penny comes up heads or tails, respectively.

1. Flip a penny ten times and move your "ant" accordingly.
2. After ten steps, report the final position of the ant, and whether it is to the right or to the left of the lamp post.
3. Again, the tally keeper puts a big X on a bar graph of the final position on the blackboard, as you did in HandsOn 10 (Figure 3.2).

Q3.14: How does your result compare with the results of HandsOn 10, in which 10 pennies are flipped at once? Is there a relationship between these two activities? Notice that in this present activity the final number of steps is equal to the number of heads minus the number of tails).

END ACTIVITY

SimuLab 4: One-Dimensional Random Walk

Now do the same activity using the computer.

The 1-D random walk is also available as a Java applet.

1. Bring up the **Random Walk** program and choose **Rand Walk** from the **New** box on the control panel.

2. Take one step at a time by pressing the **Flip** button.

3. Now make the computer flip coins automatically using the **Go** button.

4. Do the 10-step random walk 10 times (see Figure 3.5).

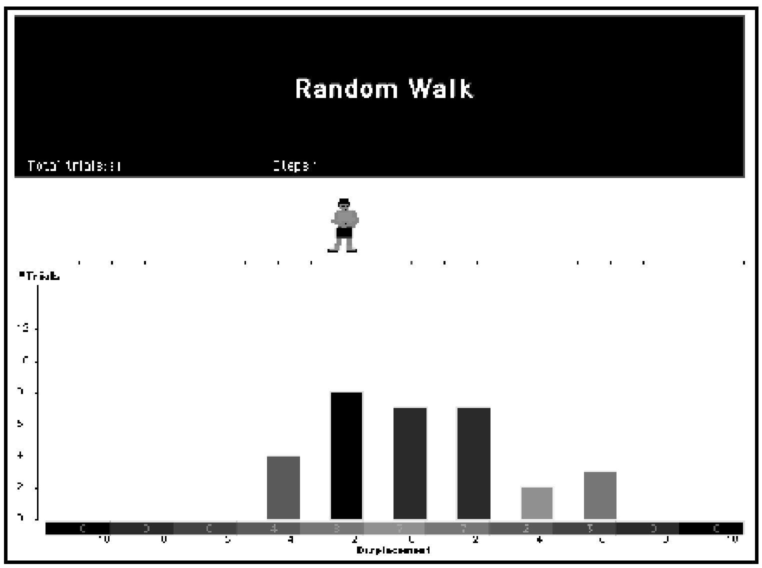

Figure 3.5: Distribution resulting from 10-step random walk.

5. Press **Coin Flip** to start a new screen, select **Less Graphics** under the **Options** menu, and press **Go**. The program will run 100 trials of 10 steps each.

6. When the 100 trials are finished, do a third 10-step random walk choosing 500 trials, under the **Experimental** menu and **Less Graphics** under **Options**.

Q3.15: Are there any similarities among the bar graphs in the three runs?

Q3.16: Does a larger number of trials make it easier to describe the shape of the resulting bar graph?

Q3.17: If you were shown only the bar graph, could you tell whether it came from the **Random Walk** part of the program or the **Coin Flipping** part of the program?

Q3.18: Why are there spaces between the bars in the bar graph for **Random Walks**, when there were no spaces between the bars for **Coin Flip**?

END ACTIVITY

Why is the random walker important? One reason the random walker is important is because it mimics the way a molecule moves. We want to study not only the motion of a single molecule, but also the motion of many molecules that combine to form the chemical and physical processes we observe every day. To do this, we need to watch many walkers at the same time.

Q3.19: Can you think of scientific or natural processes whose underlying physical or chemical dynamics could be described by a random walk?

3.4 Pascal's Triangle

Pascal's Triangle is a different way of representing coin flipping or random walk along a line. It can be pictured as a collection of pegs arranged in a triangle, as in Figure 3.6. Here instead of controlling the motion of a walker, the coin flip controls whether a ball (a marble) falling through the pegs moves to the left or right of the peg immediately below it. The same random choice is made each time the ball falls to the next lower level. Ten steps for the ant is the same as a drop of ten levels in Pascal's Triangle. The result is the same displacement right or left as in a random walk.

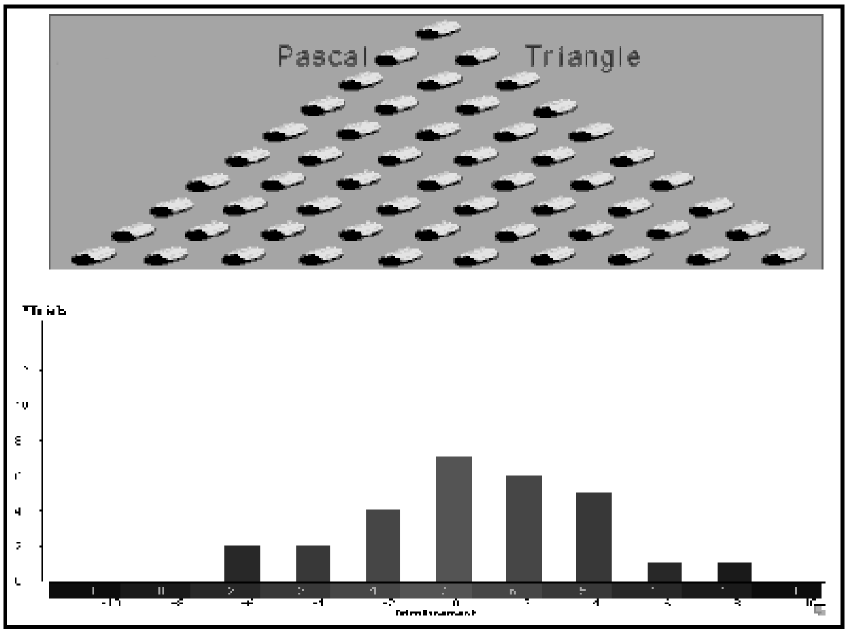

Figure 3.6: Typical distribution of many marbles after falling through Pascal's triangle.

SimuLab 5: Random Walk and Pascal's Triangle

Carry out the following steps with the **Random Walk** program.

1. Choose **Pascal's Triangle** from the **New** box on the control panel.
2. Start by doing 100 trials with 10 steps. (If you wish, select **Less Graphics** under **Options**.)
3. Select **Tile Windows** under **Options** to place the resulting bar graph in one corner of the screen (see 3.6).

4. Start the coin-flipping experiment, run 100 trials, then display the results in a second tiled window to place it next to the Pascal's Triangle results.

5. Finally bring up the **1D Random Walk** program, start it doing 100 trials, and show the resulting display in a third tiled window.

> Q3.20: Compare the graphs in the three displays. Are they identical? similar?

> Q3.21: If you ran any of the programs twice in a row, would you get the same result each time? Similar results each time? If so, what do you mean by "similar?"

> Q3.22: Choose your favorite among the three "experiments" of the **Random Walk** program (**Random Walk** or **Pascal's Triangle**) and run some trials that help you answer the following questions: How likely is it that a walker will be at least four spaces away from its starting point (right or left) after taking only four steps? After taking 8 steps? After taking 12 steps? We need to figure out how this likelihood (probability) changes as the number of steps changes.

Under the **Options** menu there is a command called **Graph Displacement**. For **Random Walk** and **Pascal's Triangle** this inserts a small graph in the lower right corner of the screen, which you can move around like any other window. On this graph, the horizontal axis shows the number of steps and the vertical axis displays the *square* of the average distance the walker is from the center after that number of steps. A green straight line shows the average slope of these data points.

Q3.23: Are the dots more scattered at the beginning of a run or at the end?

Q3.24: Are the dots more scattered at the end of a 10-trial run or at the end of a 100-trial run?

Q3.25: If you ran 30,000 trials, *guess* what the value of the slope would be?

Q3.26: What is going on? Usually when you move in a straight line your distance itself (not its square) increases linearly with time. Is this case different? If so, how and why?

END ACTIVITY

SimuLab 6: Width of a Distribution

Instead of watching one walker at a time, let's watch many walkers at the same time. Bring up the **ManyWalkers** program. In this program you choose the number of walkers displayed, from one walker to 250 walkers. When you press the **Step** button, each of these walkers takes a step randomly, either to the right or to the left. The number of walkers in each position (left or right of the origin) is shown in the bar graph at the bottom of the screen.

Q3.27: Try the display for different numbers of walkers. How do the results for 40 walkers differ from the results for 250 walkers? for 1 walker?

Q3.28: Do more walkers result in a *wider spread*, for the same number of steps? Do more walkers result in a *smoother* graph? *Guess:* How many walkers do you think it would take to yield a perfectly smooth graph?

Q3.29: How does the width of the spread change as the number of steps changes? Is the width after 12 steps three times the width after 4 steps?

END ACTIVITY

SimuLab 7: Average Position after N Steps

How far from the origin does the wandering ant end up after some number (N) of steps? What do you guess: after 10 steps is the walker more likely to be to the right or to the left of its starting point? If another ant takes 10 steps from the starting point, then another ant, then another ant, what do you expect their average final position to be after 10 steps?

Investigate! Go back to the **ManyWalkers** program and observe the value of "AVG. x" given at the right of the bar graph. The symbol x stands for the number of steps a walker is displaced from the initial position. The value of x is positive if displacement is to the right, negative if to the left. How does this average change as the number of steps increases? Is AVG. x bigger for more walkers? Or is it smaller for more walkers? Press the **Store** button to record any interesting set of averages; then press the **Table** button to examine the data you have saved.

What does theory have to say about the value of the average position of many random walkers? Here we are not talking about averaging a small number of walkers: not 100 walkers, not 1000 walkers, not even one million walkers. We ask: What is the average position of an *infinite* number of walkers? Answer: The value of the average position is zero, the position of the starting point (the lamp post)! How can this be? One word gives the reason: *Symmetry!* In this case *symmetry* means that moving right is just as likely as moving left. After any fixed number of steps, the walker is equally likely to be to the right of the starting point (positive displacement) as to the left (negative displacement). Moreover, for a given number of steps, the distance x from the starting point is likely to have the same value, whether displacement is to the right (positive x) or to the left (negative x). In brief, averaged over many millions of trials, the positive displacements cancel the negative displacements. Therefore we expect the average of many trials to be zero; the *average* position of the walkers is at the starting point.

The *average* position may be zero, but the *spread* of positions is not zero, as shown also by the examples in Figures 3.5 and 3.6. The number of heads in 10 trials does not always come out the same. The number of heads can vary in each 10-step trial.

How can this spread of final positions be described? Pascal's Triangle helps to answer this question. Look at the Pascal Triangle shown in Figure 3.7. We are going to do something that turns out to be very useful: Count the number of paths that can lead to each of the pegs in the 2nd row, the 3rd row, the 4th row, etc.

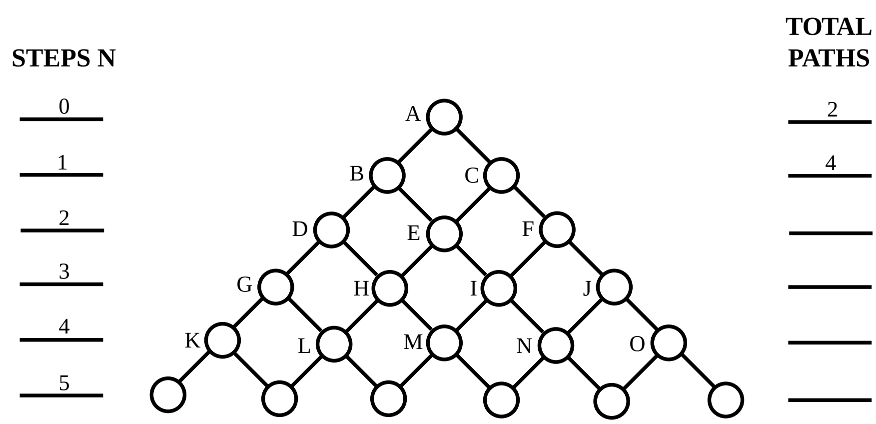

Figure 3.7: Pascal's Triangle showing the number of paths by which the falling marble can arrive at each peg. This one is for a 5-step random walk.

The first two rows are done already. For example, the only way to get to point D (after 2 steps) is by going left-left (LL). We enter a 1 in circle D, meaning there is only one way to get there. In contrast, there are two alternative paths to point E, namely left-right (LR) or right-left (RL). Therefore we enter the number 2 in circle E. How many paths are there to point F? Now you fill in the next two rows. You need 3 steps and 4 steps to reach the pegs in these rows, respectively.

Now we state a very important rule. Starting from some initial point, and for a random choice at each step . . .

every alternative path to a given final point is equally likely.

In other words, two paths are *alternative*—or *different*—if *any* segment of one path is different from that of the other path.

As an example of the new rule, look at row 3, where the ball arrives after two steps. There are two paths to central position E, but only one path to each of the end positions D and F. The rule above says that a random walker is twice as likely to end up at the central position E than at either of the positions D or F.

Is this rule reasonable? We have assumed that as the ball comes to each peg, it is equally likely to fall to the left of the peg as to the right. And every possible path is made up of a sequence of these equal choices. So it is reasonable that every path to a given final point is as likely as every other possible path to that point.

It is easy to find a rule that allows us to determine how many (equally likely!) paths lead to a given peg in Pascal's Triangle: The number in each circle is the sum of the numbers in the two adjacent circles above it in the previous row. (For the end circles, the number is the same as in the *one* adjacent circle in the row above.) For example, the number in circle E is 2, equal to the sum of the numbers $1+1$ in circles B and C above it. Is this reasonable? Think of paths. There is 1 possible path leading into B, one possible path leading into C. Therefore they provide

$$1 + 1 = 2 \text{ possible paths}$$

to circle E below them.

Use this rule to make entries in the circles of rows 3 and 4 in a copy of Figure 3.7. Also put numbers in the circles for the fifth row at the bottom of the figure.

If every alternative path is equally likely, then the more paths that lead to a given peg, the more likely the ball is to arrive at that peg. For example, two paths lead to peg E in Figure 3.7, while only one path leads to each of pegs D and F in the same row. The total number of paths that arrive in that row are $1+2+1=4$ (number at the right side of the figure). The probability that a ball arrives at peg E is $2/4 = 1/2$. A similar analysis can be applied to a peg in any row of Figure 3.7.

Using Pascal's Triangle, we have described the number of different paths (and relative likelihood, that is, probability) of arriving at any location (at any circle in the diagram) after a certain number of steps. The result is too many numbers. It is time to simplify our picture by using averages.

END ACTIVITY

3.5 Measuring Average Distances

Starting at the lamp post, the ant wanders a certain number of steps, randomly, to the right and to the left. The ant records its final distance from the lamp post: positive displacement measured to the right, negative displacement measured to the left. Then for the second trial the ant goes back to the lamp post and randomly takes the same number of steps again, recording its final position. Then a third trial, then a fourth trial, and so on. Finally the ant calculates the *average* final displacement from the lamp post from all the trials. What do you expect the average displacement to be?

We already know the answer: This average final position is zero, the starting point. This answer is verified by the numbers in the circles of Pascal's Triangle (Figure 3.7). For every row (each row representing the expected displacements after a given number of steps) the numbers in the circles are the same to the right of the initial position (positive final displacement) as they are to the left of the initial position (negative final displacement). In taking the average, final displacements to the right are typically canceled by final displacements to the left—by symmetry!

Thus zero is the average displacement of the random walker after many trials, no matter how many steps the walker takes. Yet by experimenting we know that the *spread* of final positions increases with the number of steps. The number of final positions available increases as the number of steps increases in Pascal's triangle. That is why the triangle is wider at the bottom. Notice that after two steps, 2 of the total of 4 possibilities leaves the ant at its starting point. That's a 50% chance. After 4 steps, 6 of the total of 16 possibilities leaves the ant at the starting point—a decrease to 37.5%.

> Q3.30: Q30: After 8 steps, what percentage of the possibilities leaves the ant at its starting point? What other calculations can we consider that will help us to understand the spread of final positions?

There are various ways to measure this spread. We would like to get around the fact that rightward and leftward displacements tend to cancel one another. One possibility is to average the *absolute values* of the displacements. An absolute number is never negative; therefore when we average the absolute displacements, we will get a result that is positive (or perhaps zero). This leads to the idea of an *average absolute displacement*.

SimuLab 8: Measures of Average Squared Displacement

Return to the **ManyWalkers** program. This time pay attention to the value of "AVG. $|x|$" given at the right of the bar graph. The symbol $|x|$ stands for "absolute value of x," or "magnitude of x."

> Q3.31: Does AVG. $|x|$ increase with the number of steps?

> Q3.32: Q32: For a given number of steps, does AVG. $|x|$ have a larger value for more walkers?

The average absolute displacement is not the measure chosen by scientists to describe the random walker, because it does not give the simplest result, as will be shown below. The result is simpler if we take the *square* of each displacement and then average these squares. The square of a number is positive, even when the number itself is negative. (If the number is zero, its square is also zero.) Therefore the average of squares of final displacement will never be negative. This average is called the *average squared displacement* or *mean square displacement*.

Here are the results of an experiment in which 20 ants each took 3 steps:

Number of ants	Final displacement
2	- 3
9	- 1
7	1
2	3

To find the average square displacement we calculate as follows:

- 2 ants had a squared displacement of $(-3)^2$ or 9
- 9 ants had a squared displacement of $(-1)^2$ or 1
- 7 ants had a squared displacement of $(1)^2$ or 1
- 2 ants had a squared displacement of $(3)^2$ or 9

Averaging we get

$$\frac{2(9) + 9(1) + 7(1) + 2(9)}{20} = 2.6$$

for the average squared displacement. (The average number of steps does not have to be an integer; if one takes one step and another takes two steps, the average is 1.5 steps.) This result will naturally be a bit different for each trial.

Return to the original picture of the wandering ant (Figure 3.4).

1. Start with the ant in the center.
2. Flip a coin and move the ant one step.
3. Record its position (+1 or -1) in a copy of Table 3.2.
4. Now flip the coin again, move the ant, and record its new position.
5. Continue for a total of five steps, recording the ant's position after each coin flip.

6. Now square the total distance (displacement) from the starting point after each coin flip.

7. We want to graph the average squared displacement *versus* the number of steps. Plot your data in a distinctive color on a graph with number of steps along the horizontal axis and x^2 along the vertical axis, where x is the displacement.

8. Repeat STEPS 1 through 7 using a second ant, again recording the position after each coin flip.

9. This time take the average of the squared displacements of the two walkers and plot this in another color (green perhaps) on the graph.

10. Continue with the third walker, this time taking the average of the squared displacements of all three walkers after each coin flip. Plot this in yet another color (maybe blue).

Table 3.2: Computing average of the squared displacement of the random walker.

	Walker One		Walker Two			Walker Three		
Step	$x =$	$x^2 =$	$x =$	$x^2 =$	Ave x^2 of walkers #1 and #2	$x =$	$x^2 =$	Ave x^2 of walkers #1, #2 and #3
1								
2								
3								
4								
5								

Q3.33: As more walkers are added, does the graph of the resulting averages approach a pattern? Using yet another color (black?), draw a line that shows this pattern.

Can we make any prediction about the value of the average squared displacement after many trials? Once more, we can use the computer to give us many trials.

1. Bring up the **ManyWalkers** program and look at the value of "AVG. x^2" at the right of the bar graph.

2. Try different numbers of walkers and different numbers of steps.

> Q3.34: Does AVG. x^2 increase with number of walkers, for a given number of steps? In contrast, does AVG. x^2 increase with the number of steps, for a given number of walkers?

3. **Store** the data on the table and examine the **Table**.

4. Call up the **Graph**.

> Q3.35: Do you notice anything which might help you predict the value of AVG. x^2? In particular, can you predict the value of AVG. x^2 for 250 walkers after 30 steps?

> Q3.36: Print out the graph for 250 walkers taking 30 steps. Then run 40 walkers taking 30 steps and print out the resulting graph. Finally, run 10 walkers taking 30 steps and print out the graph. Label these three graphs, lay them side by side, and compare them. What accounts for the differences and similarities between these three graphs?

Predict what you expect the graph to look like when *one* walker takes 30 steps. Try it and compare the result with your prediction.

You can watch the change in the graph as the number of trials increases. To do this, return to the **Random Walk** program and open the **Graph Displacement** window.

END ACTIVITY

3.6 Proving Average Squared Distance (Optional)

What is the value of the average square displacement of a random walker after N steps? Here we show two ways to calculate this average square displacement.

First Method: Average Square Displacement from Pascal's Triangle

Look at Pascal's Triangle, Figure 3.7.

Pick out the row of circles representing displacements after two steps. A total of 4 paths are available for entering this row (the denominator of the average in the equation below). Two paths lead to zero displacement, one path leads to a displacement $+2$, and one path leads to a displacement -2. Each of these paths is equally likely. In taking the average of the squares (numerator of the equation below), there is one entry for $(-2)^2 = 4$, two entries for $(0)^2 = 0$, and one entry for $(+2)^2$. Therefore the average of the square for two steps is:

$$\frac{1(-2)^2 + 2(0)^2 + 1(+2)^2}{4} = \frac{8}{4} = 2.$$

Now calculate the average of the squares of the displacements after three steps. A total of 8 paths are available for entering this row. Three paths lead to $+1$ final displacement. Three paths lead to -1 final displacement. One path leads to $+3$ and one path to -3 displacement. Follow the same steps as above to calculate the average of the squares of the displacements after three steps:

$$\frac{1(-3)^2 + 3(-1)^2 + 3(+1)^2 + 1(+3)^2}{8} = \frac{24}{8} = 3.$$

Show that for the row representing displacement after four steps the average of the squares of the displacements is 4. What is the average of the squares of the displacements after five steps?

Do you see a pattern? The average of the squares of the final displacements is equal to the number of steps. A graph of the ideal mean square displacement vs. the number of steps is simply a straight line.

Second method: Calculating the Average Square Displacement using Algebra

Notation: Scientists often use the symbol x to represent displacement along a line, x^2 to represent the square of this displacement, a subscript N to represent "after N steps," and a bracket $\langle\ \rangle$ to represent average value. Then our argument from Pascal's triangle is that:

$$\text{(average of the squared displacements after } N \text{ steps)} = \text{number of steps } N,$$

which is written

$$\langle (x_N)^2 \rangle = N.$$

We already obtained this result using Pascal's Triangle. Next we calculate the same answer using algebra.

Suppose the walker has taken n steps and is now at position x_n. What do we *expect* the value of $\langle (x_n)^2 \rangle$ to be? Start by asking where the walker will be at the next step $n+1$. If we know where the walker is now (i.e., x_n) then after the next step the walker can be a step to the right or a step to the left; either at

$$x_{n+1} = x_n + 1 \qquad \text{(step right)} \tag{3.1}$$

or at

$$x_{n+1} = x_n - 1 \qquad \text{(step left).} \tag{3.2}$$

Which of these will it be? We cannot say for sure. In a random walk both are equally likely. So we take an average: Square both sides of Eqs. (3.1) and (3.2) and take the average of the two. Again, use the $\langle\ \rangle$ bracket to mean average value. Then

$$\langle (x_{n+1})^2 \rangle = \frac{\langle (x_n+1)^2 + (x_n-1)^2 \rangle}{2} = \frac{\langle x_n^2 + 2x_n + 1 + x_n^2 - 2x_n + 1 \rangle}{2},$$

or

$$\langle (x_{n+1})^2 \rangle = \frac{2\langle (x_n)^2 \rangle + 2}{2},$$

or

$$\langle (x_{n+1})^2 \rangle = \langle (x_n)^2 \rangle + 1. \tag{3.3}$$

What does this equation mean? Start with $n = 0$, the zeroth step (or no step at all). Then $\langle (x_0)^2 \rangle = 0$. For the first step, Eq. (3.3) tells us that $\langle(x_1)^2\rangle = \langle(x_0)^2\rangle + 1 = 0 + 1 = 1$, which we knew already without doing this calculation. From this, it follows that $\langle(x_2)^2\rangle = \langle(x_1)^2\rangle + 1 = 1 + 1 = 2$ and $\langle(x_3)^2\rangle = 3$ and, in general $\langle(x_N)^2\rangle = N$. The result? The average squared displacement after N steps is simply N:

$$\langle(x_N)^2\rangle = N.$$

This is the same result we obtained from studying Pascal's Triangle.

What we stated above is an ideal average, i.e., an average we would also obtain over an infinite number of trials. For a finite number of trials, for example the average of 250 walkers, this is our "best guess" of the final squared displacement after N random steps along a line. In general, observed values approach the "best guess" for a very large number of trials.

3.7 The Wandering Ant on a Square Grid

Suppose that the ant is not forced to step just along a line, but can move in four mutually perpendicular directions when walking away from the lamp post. This type of movement is called a *2-dimensional random walk*.

For example, an ant is standing in the center of a 11 by 11 grid, as shown in Figure 3.8. Each grid square is the size of one step. The ant can move one step at a time in one of four directions: north, south, east, or west. The ant cannot move diagonally or take more than one step at a time. If the ant walks off the edge of the grid, it cannot return.

Q3.37: Where do you think the ant will most likely be after 10 steps? Will it still be on the grid?

Q3.38: Where do you think the ant will most likely be after 100 steps? Will it still be on the grid?

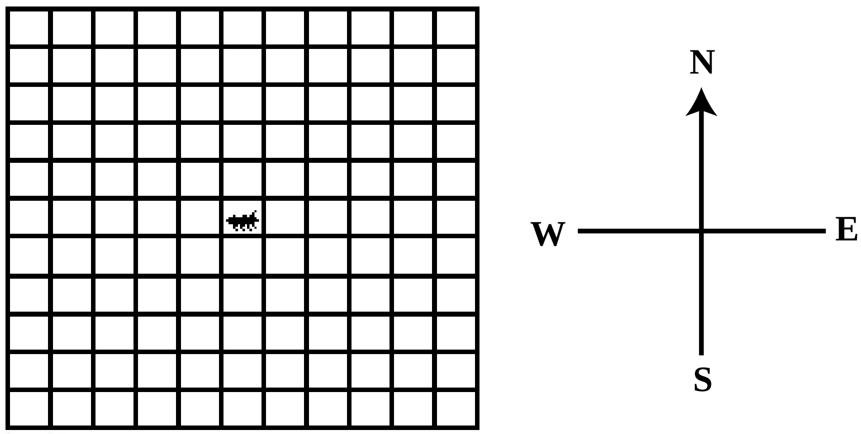

Figure 3.8: The wandering ant in a 2-dimensional random walk.

Q3.39: Let's say we place 1000 ants on the center square of the grid. If each ant moves independently using the same rules as above, how do you think the ants will be distributed on the grid after each ant has taken 10 steps? 100 steps? 1000 steps?

Q3.40: Is there a relationship between random walks and coin flipping? If so, what is this relationship?

HandsOn 12: Random Walk in 2-Dimensions

For the following activity you will need:

- a checkerboard and one checker,
- two copies of the grid in Figure 3.8, and
- a 4-sided die (or use a regular 6-sided die and throw again when the result is a 5 or 6).

) and 2-D dom walk a applets available.

Place a checker in the center of a checkerboard. Flip a four sided die labeled north, south, east and west and move the "ant" accordingly. After 10 steps, mark on a copy of the checkerboard the final position of the random walker. Start again from the center and repeat the same 10-step procedure ten times.

Q3.41: Measure the distance from the origin for each random walker after 10 steps and take the average of all the distances. If several groups are doing the same activity, average your averages. What is your result? Compute the square of the distance from the origin for all walkers and take the average. What is your result?

END ACTIVITY

SimuLab 9: The Deer Program and Population Dynamics

The **Deer** program uses the 2-dimensional random walk to model the variations that occur in a deer population. A deer moves randomly on a field, eating the grass wherever it steps. Grass grows back in each eaten square after a certain amount of time. (Time is measured in steps: in one time step, every deer takes one random step.)

Will the deer population grow or decline? A deer dies if it takes a specified number of steps in a row onto squares where the grass has been eaten and has not yet grown back. Every live deer has an offspring after another specified number of steps.

1. Bring up the **Deer** program. On the left you should see the control window, on the right a green field.

2. After reading the introduction under **Describe**, create a single deer by clicking the cursor in the middle of the field.

3. Start the program by clicking on **Go**.

4. Select **Show Graph** under the **Control** menu. The top portion of the graph shows the number of deer present in the field; the bottom portion the percentage of grass still alive on the field.

5. If at any time you want to stop the program, press the **Pause** button.

Q3.42: What happens to the size of your deer population? Does it grow and grow? Does it increase, then decrease? Does it die out?

Q3.43: Suppose you started again with a deer in exactly the same position. Would the population at every step of the second run be exactly the same as the first? Would the final result be the same?

Now you are going to place a dozen deer in the field and let the program run as before. Before doing this, write down your *predictions* about what will happen:

Will the initial population growth be different for 12 deer than for one deer? Will the final outcome be different for 12 deer than for one deer?

Now start the program as before, but this time click at a dozen different points on the field. Then start the program by clicking outside the field.

Q3.44: What happens to a deer that wanders off, say, the right-hand side of the field?

Q3.45: Q45: Look back at your predictions for 12 deer. Were your predictions correct?

END ACTIVITY

3.8 Models in Science

In this book, the words *science*, *scientist*, and *scientific* added together appear only a total of 80 times. On the other hand the word *fractal* appears 160 times, and the word *model* a whopping 250 times. Apparently the word "model" is of key importance when we do science.

In Chapter 1 we read "Now we use a computer to draw the *model* of a coastline and to measure the dimension of that *model*," and "...you created a *model* of a fractal coastline using a rubber band, thumb tacks, and a die (or a rope, a die, and a coin)." In Chapter 2 (this chapter) we have used a simple, one-dimensional random walk *model* to understand random motion in general. As we progress through the book we will encounter the word again and again:

- "Other random fractal patterns also have dimensions between 1 and 2: a snowflake, a nerve cell, a lightning stroke. The growth of these structures can be *modeled* by a process called aggregation ... In the following activity you will use a 2-dimensional random walk to mimic (*model*) the aggregation process." (Chapter 3)

- "Any good *model* is much simpler than the phenomenon, but it reproduces the essential features of the phenomenon ... understanding in science grows as we progress from *model* to *model*." (Chapter 4)

- "In order to build a mental *model* to analyze the bacteria experiment, you need certain facts about bacteria and their behavior." (Chapter 5)

This is only a sampling of what appears throughout this book. Similar examples exist in every chapter. Generally speaking, *the purpose of a model is to simplify reality so that reality can be analyzed*. If a model is only partially successful in predicting behavior, we attempt to modify the model and improve its assumptions so that its predictions will be more accurate. This process tends to generate models that are increasingly complex, models that provide further challenge to scientific investigation.

3.9 What Do You Think?

Q3.46: Describe in your own words the meaning of a model? Can you think of examples of how models are used to describe nature?

Q3.47: In this chapter we have used a very simple model: an ant wandering back and forth with steps of equal length taken at equal time intervals. Yet this simple model describes many processes in the real world. How can this be, since our model is so simple? Very similar results are predicted by more complicated models that add more randomness: steps of random length, steps in random directions, steps that take place randomly in time. It turns out that the predictions of these more complicated models are similar to ours as long as our model reflects the *average* step length, *average* time between steps, and *average* distance from the starting point. Often in science a simple, easily understood model makes good predictions about the more complicated real world.

Q3.48: Does changing the number of steps the random walk takes effect the relation between "mean squared distance" and "step number"? Does it change the "average distance" from the origin?

Q3.49: Why would we care about being able to relate "mean squared distance" and "step number"? What does this tell us? Hint: think in terms of "predicting".

Q3.50: Can you think of examples from nature where particles may move around in a random way? List examples from nature where the random motion is biased. Can you list some of the possible sources of the bias (e.g., draft air currents would bias the smell of the unstoppered ammonia bottle)?

Research Projects

Research projects can be published on our Web site.

Try the suggestion below, design your own, or write an essay using any of the questions throughout this chapter as inspiration.

Suggested Project: Bring up the **Deer** program again. Change the parameters in the **Control** menu.

> The **Size of a Deer** (in screen pixels) determines how big the field is: **Smaller** deer means smaller squares on a field of constant size, and so a field that can accommodate more steps, more grass plots, and more deer.
>
> A deer dies if it fails to step on a green square for a number of steps greater than the setting of the **Number of steps without food**.
>
> The **Time of grass restoration** is the number of steps required before grass is restored to an "eaten" square.
>
> The **Breeding age of deer** is the number of step-cycles after birth at which a live deer has an offspring. Every live deer has one offspring every time it takes this number of steps.

Your first task is to find a combination of these settings such that the population of deer reaches a constant value and stays there—more or less! This is called a *stable population*. Be systematic about the search, writing down in your notebook each setting and the population outcome. Is there more than one combination of settings that achieve this goal? If so, is there a pattern of such settings?

Your next tasks are to find (i) which changes in the settings from this stable condition result in a more or less steady *increase* in population, and (ii) which changes result in a more or less steady *decrease* in population. Is there a pattern to these changes? Can you find changes that lead to an oscillating population, one that increases and decreases rhythmically or erratically?

How would adding a predator complicate the problem and change the outcome? A predator, such as a wolf, eats deer, has baby predators, and dies if it does not meet a deer after a given number of steps.