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Multi-scaling in finance

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The most suitable paradigms and tools for investigating the scaling structure of financial time series are reviewed and discussed in the light of some recent empirical results. Different types of scaling are distinguished and several definitions of scaling exponents, scaling and multi-scaling processes are given. Methods to estimate such exponents from empirical financial data are reviewed. A detailed description of the Generalized Hurst exponent approach is presented and substantiated with an empirical analysis across different markets and assets.

Keywords: Econophysics; Scaling; Multifractal formalisms; Time series analysis

1. Introduction

The scaling concept has its origin in physics but is increasingly applied to other disciplines (Müller *et al.* 1990, Mantegna and Stanley 2000, Dacorogna *et al.* 2001a, Bouchaud and Potters 2004, Loffredo 2004). In the past, economists have looked across many different socio-economic data to find the existence of scaling laws (Brock 1999). Analogously, financial analysts have searched for patterns in financial prices that are repeated at different time scales. The best documented beginning was probably the work of Elliot in the 1930s where he emphasized the appearance of patterns at different time horizons (Frost and Prechter 1978). Until the 1960s, the only stochastic and scaling model in finance was the Brownian motion, originally proposed by Bachelier (1900), and developed several decades later (Osborne 1959). This theory predicts that the returns of market prices should follow a normal distribution with stable mean and finite variance. However, there is ample empirical evidence that the returns are not normally distributed, but have a higher peak around the mean and fatter tails (Mantegna and Stanley 2000, Dacorogna *et al.* 2001a, Bouchaud and Potters 2004). Generalizations of the classical Brownian motion were made by Mandelbrot and followers involving either fractional Brownian motion (Mandelbrot 1965, 1997, Mandelbrot and Van Ness 1968, Clark 1973), or Lévy motion (Mandelbrot 1962, 1963, 1967, Fama 1963, 1965, Mirowski 1995). The above approaches generally involve additive monofractal processes and analyses; but, in contrast, several scaling systems appear to be more complex.

In recent years, the application of the scaling concept to financial markets has largely increased also as a consequence of the abundance of available data (Müller *et al.* 1990). There has been renewed interest in cross-disciplinary research and a new field, Econophysics, has developed around some of these themes. In this framework, the earliest results on scaling laws using high-frequency foreign exchange data can be found in Müller *et al.* (1990). Along with this work, these authors produced several other interesting and important studies (Dacorogna *et al.* 2001a). Several others followed, such as Mantegna and Stanley (1995), Evertsz (1995), Ghashghaie *et al.* (1996) and Mantegna and Stanley (2000), and now the field has many examples of scaling and power laws detected in many other financial data.

In our opinion, even if the existing literature has shown that the use of tools from physical sciences is very useful to obtain a better description of financial markets, much more has to be done. Indeed, several models have been developed but they do not work too well, being over-simplifications of reality. They explain some of the empirical evidence, but leave many questions unanswered. Empirical analysis may be used to improve existing models and even make new models that conform more closely to observed market behaviour. Recently, a controversy has erupted between LeBaron (2001) on one side and Mandelbrot (2001) and Stanley and

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Plerou (2001) on the other, with Lux (2001) somewhere in the middle, to determine if the processes that describe financial data are truly scaling or if the similarity observed at different scales is simply an artifact of the data. Moreover, these papers propose new scaling models or empirical analyses that better describe the empirical evidence (and one could add to these Bouchaud *et al.* (2000)). It should be noted, however, that as underlined by Stanley *et al.* (1996), in statistical physics, when a large number of microscopic elements interact without a characteristic scale, universal macroscopic scaling laws may be obtained independently of the microscopic details. Multi-scaling processes have also been used in many contexts to account for the time-scale dependence of the statistical properties of a time series. Recent empirical findings suggest that, in finance, this framework is likely to be pertinent (Ghashghaie *et al.* 1996, Calvet and Fisher 2002) and there are several multi-scaling models proposed in the literature (Ding *et al.* 1993, Muzy *et al.* 1994, 2000, Ghashghaie *et al.* 1996, Mandelbrot *et al.* 1997, Arneodo *et al.* 1998, Mandelbrot 1999, Bacry *et al.* 2001, Calvet and Fisher 2001, Muzy and Bacry 2002, Lux 2003, 2004, 2006, Eisler and Kertesz 2004, Borland and Bouchaud 2005, Borland *et al.* 2005). Di Matteo *et al.* (2003, 2005) address the question of the scaling properties of financial time series from another angle and the multi-scaling properties of several financial markets are analysed and compared. In particular, different markets, very developed as well as emerging markets, have been studied in order to see if the scaling properties differ between the two and if they can serve to characterize and measure the development of the market. By means of relatively simple statistics (very much along the lines of the review paper by Brock (1999)), indications on the market characteristics have been obtained without fitting any new model, but only by gathering empirical evidence. The studies of Di Matteo *et al.* (2005) were motivated and inspired by the following motivations.

1.1. Motivation

Scaling-type regularities in data give useful information on the underlying data generating process. The challenge for empirical and theoretical researchers lies in uncovering what these scaling law regularities tell us about the underling mechanisms that generate the data and in using the empirical scaling evidence as a ‘stylized fact’ that any theoretical model should also reproduce.

In a recent book (Dacorogna *et al.* 2001a), the hypothesis of heterogeneous market agents was developed and backed by empirical evidence. According to this view, agents are essentially distinguished by the frequency at which they operate in the market. Scaling analysis, which looks at the volatility of returns measured at different time intervals, is a parsimonious way of assessing the relative impact of these heterogeneous agents on price movements. Viewing market efficiency as a result of the interaction of these agents (Dacorogna *et al.* 2001b) brings us naturally to believe that it is the presence of many different agents that would characterize a mature market, while the absence of some type of agents should be a feature of less-developed markets. Such a fact should then be reflected in the measured scaling exponents. Study of scaling behaviour must therefore be an ideal candidate to characterize markets.

For institutional investors, a correct assessment of markets is very important to determine the optimal investment strategy. It is common practice to replicate an index when investing in well-developed and liquid markets. Such a strategy minimizes the costs and allows the investor to fully profit from the positive developments of the economy while controlling the risk through the long experience and the high liquidity of these markets. When it comes to emerging markets, it is also clear that stock indices do not fully represent the underlying economies. Despite its higher costs, an active management strategy is required to control the risks and fully benefit from the opportunities offered by these markets. The differentiation between markets is clear for the extreme cases: the New York stock exchange and the Brazilian or Russian stock exchange. The problem lies for all those in between: Hungary, Mexico, Singapore and others. For these markets a way of clarifying the issue will help to decide on the best way to invest assets. Until now, most work has concentrated on studies of particular markets: Foreign Exchange (Müller *et al.* 1990, Corsi *et al.* 2001, Dacorogna *et al.* 2001a), the US Stock Market (Dow Jones) (Mantegna and Stanley 1995) or Fixed Income (Ballocchi *et al.* 1999). These studies showed that empirical scaling laws hold in all these markets and for a large range of frequencies: from a few minutes to a few months. Di Matteo *et al.* (2005) report a study that, to our knowledge, represents the widest empirical investigation across 32 different markets that deal with different instruments: equities, foreign exchange rates and fixed-income futures.

This short review is structured in the following way. In section 3 we report some fundamental concepts used in this paper and in other papers dealing with scaling analysis. In sections 4, 5, 6 and 7 we review several techniques and estimators proposed and used for investigating scaling properties and the Hurst exponent. In particular, section 5 describes the generalized Hurst exponent in the multi-scaling framework. An application of this multi-scaling method to financial market data is given in section 8. Finally, conclusions are given in section 9.

Before starting with the core topic of this work, let us clarify an important aspect reported in the following section.

2. Two different types of scaling in finance

There are two types of scaling behaviour studied in the finance literature.

1. The behaviour of some forms of volatility measure (variance of returns, absolute value of returns) as a

function of the *time interval* on which the returns are measured. (This study will lead to the estimation of a *scaling exponent* related to the fractal dimension and to the Hurst exponent.)

2. The behaviour of the tails of the distribution of returns as a function of *the size of the movement*, but keeping the time interval of the returns constant. (This will lead to the estimation of the *tail index* of the (Dacorogna *et al.* 2001a).)

Although related, these two analyses lead to different quantities and should not be confused as is often the case in the literature (LeBaron 2001, Lux 2001, Mandelbrot 2001). For a further explanation of this and the relation between the two quantities, the reader is referred to the excellent paper by Groenendijk *et al.* (1998). The empirical results reported in section 8.1 refer to the first type of analysis (Di Matteo *et al.* 2003, 2005).

3. Several definitions

This section recalls some basic concepts such as a self-affine process, the fractal dimension D, the Hurst exponent H and its relation to D, the definition of a multi-scaling process, the Hölder exponent and the multi-fractal spectrum.

3.1. Self-affine process

A transformation is called affine when it scales time and size by different factors, while behaviour that reproduces itself under an affine transformation is called self-affine (Mandelbrot 1997). A time-dependent self-affine process $X(t)$ has fluctuations on different time scales that can be re-scaled so that the original signal $X(t)$ is statistically equivalent to its re-scaled version $c^{-H}X(ct)$ for any positive c, i.e. $X(t) \sim c^{-H}X(ct)$. More formally,

Definition 3.1 A random process $\{X(t)\}$ that satisfies

$$\left\{X(ct_1), \ldots, X(ct_k)\right\} \stackrel{\mathrm{d}}{=} \left\{c^H X(t_1), \ldots, c^H X(t_k)\right\}, \quad (1)$$

for some $H > 0$ and all $c, k, t_1, \ldots, t_k \geq 0$ is called self-affine.† H is the self-affinity index, or scaling exponent of the process $\{X(t)\}$. Brownian motion, the L-stable process, and the fractional Brownian motion (Feller 1971) are the main examples of self-affine processes in finance. As suggested by Mandelbrot (1963) the shape of the distribution of returns should be the same when the time scale is changed. On the contrary, empirical evidence shows that many financial series are not self-affine, but instead have thinner tails and become less peaked in the bells when the sampling interval increases (Calvet and Fisher 2002). Therefore, there is a need to consider a more complex type of process.

3.2. Fractal dimension and Hurst coefficient

Let us start from the definition of the fractal dimension, D, and Hurst coefficient, H, for the simplest case concerning a stationary standard Gaussian random function $X(t)$ with $E(X(t)) = 0$ and $E(X^2(t)) = 1$, $E(\ldots)$ being the expectation value. The function $X(t)$ defines a profile in the Euclidean plane and its autocorrelation function (Taqqu 1981, Mainardi *et al.* 2000, Scalas *et al.* 2000, Embrechts and Maejima 2002, Bassler *et al.* 2006),

$$C(\Delta t) = E[X(t)X(t + \Delta t)], \quad (2)$$

is a measure of the profile roughness. A fractal dimension can be defined if the correlation function behaves as

$$C(\Delta t) \sim 1 - |\Delta t|^{\alpha}, \quad \text{as } \Delta t \to 0, \quad (3)$$

for $\alpha \in (0, 2]$. In this case, one can associate with such a function the fractal dimension

$$D = 2 - \frac{\alpha}{2}. \quad (4)$$

On the other hand, the asymptotic behaviour at infinity ($\Delta t \to \infty$) quantifies the presence or absence of long-range dependence, and if $C(\Delta t)$ behaves as

$$C(\Delta t) \sim |\Delta t|^{-\beta}, \quad \text{as } \Delta t \to \infty, \quad (5)$$

for $\beta \in (0, 1)$, then the process has a long memory with Hurst coefficient

$$H = 1 - \frac{\beta}{2}. \quad (6)$$

The notions of D and H are closely linked and often confused in much of the scientific literature (Gneiting and Schlather 2004). Generally speaking, the fractal dimension (Stanley 1971, Feder 1988, Barabasi and Stanley 1995, Mandelbrot 1997) of a profile or surface is a roughness measure, with $D \in [n, n+1)$ for a surface in R^n, with higher values indicating rougher surfaces. Long-memory dependence or persistence in time series or spatial data are instead associated with power-law correlations and often referred to as Hurst effects. Long-memory dependence is characterized by the Hurst coefficient H. The two quantities D and H are independent of each other: the fractal dimension is a local property, and long-memory dependence is a global characteristic. However, for self-similar processes, the local properties are reflected in the global properties. The relationship

$$D = n + 1 - H \quad (7)$$

between D and H holds for a self-similar process in n-dimensional space. Long-memory dependence, or persistence, is associated with the interval $H \in (0.5, 1)$ and is therefore linked to low fractal dimensions. Rougher processes with higher fractal dimensions occur

†In the literature, self-affine processes are also called self-similar.

for antipersistent processes and result in coefficients $H \in (0, 0.5)$. But, in this case, H is the self-affine index defined in equation (1). The self-affine index is often identified with the Hurst coefficient, but this identification is appropriate only if $H \in (0.5, 1)$ (Gneiting and Schlather 2004). The above definitions and the associated linear relationship between D and H (equation (7)) formally hold only for a Gaussian process; however, they are believed to be valid for a large number of real-world data sets. Let us stress that much care should be taken in verifying the validity of the self-similarity assumption.

3.3. *Multi-scaling process*

A stochastic process $\{X(t)\}$ is called multi-scaling if it has stationary increments and satisfies (Calvet and Fisher 2002)

$$E(|X(t)|^q) = c(q)t^{\tau(q)+1}, \tag{8}$$

for all $t \in \mathcal{F}$, $q \in \mathcal{L}$, with $\mathcal{F}$ and $\mathcal{L}$ intervals on the real line ($\mathcal{F}$ and $\mathcal{L}$ have positive lengths, and $0 \in \mathcal{F}$, $[0, 1] \subseteq \mathcal{L}$), $\tau(q)$ and $c(q)$ functions with domain $\mathcal{L}$, and $E(\ldots)$ the expectation value. The function $\tau(q)$ is called the scaling function of the multi-scaling process and it is concave. From equation (8) we see that all scaling functions must have the same intercept at $q=0$: $\tau(0) = -1$. Linear scaling functions $\tau(q)$ are determined by a unique parameter (their slope) and the corresponding processes are called uniscaling or unifractal. Let us prove here that a self-affine process $\{X(t)\}$ is multi-scaling and has a linear function $\tau(q)$. Denoting by H the self-affinity index introduced before, we observe that the invariance condition $X(t) = t^H X(1)$ implies that $E(|X(t)|^q) = t^{qH} E(|X(1)|^q)$. Equation (8) therefore holds with $c(q) = E(|X(1)|^q)$ and $\tau(q) = qH - 1$. In this special case, the scaling function $\tau(q)$ is linear and fully determined by its index H. Uniscaling processes, which may seem appealing for their simplicity, are not, however, satisfactory models for asset returns. This is because most financial data sets have thinner tails and become less peaked in the bell when the sampling interval, Δt, increases. In section 8.1 we will show empirical evidence for financial data that have nonlinear $\tau(q)$.

3.4. *The Hölder exponent and the multi-scaling spectrum*

The local Hölder exponent, $\alpha(t)$, quantifies the scaling properties of the process at a given point in time, and is also called the local scale of the process at t. If we consider a stochastic process $X(t)$, its infinitesimal variation around time t is

$$|X(t + \mathrm{d}t) - X(t)| \sim C_t(\mathrm{d}t)^{\alpha(t)}, \tag{9}$$

where $\alpha(t)$ and C_t are, respectively, the local Hölder exponent and the prefactor at t. The Hölder exponent thus describes the local scaling of a path at a point in time, and smaller values correspond to more-abrupt variations. A unique scale $\alpha(t) = 1/2$ is observed on the jagged sample paths of a Brownian motion. Similarly, a fractional Brownian process is characterized by a unique exponent $\alpha(t) = H$. The uniscaling process is characterized by one single Hölder exponent, whereas multi-scaling processes contain a continuum of local scales and the Hölder exponent is not unique. In this case, a multi-fractal spectrum $f(\alpha)$ can be defined and it can be interpreted as the limit of the normalized histogram of Hölder exponents (Mandelbrot 1997). Therefore, the multi-scaling of a certain process or data set is reflected in the existence of a continuum of Hölder exponents, while uniscaling processes would be characterized by a degenerate spectrum: a unique Hölder exponent H.

4. Re-scaled range statistical analysis

The scaling properties in time series have been studied by means of several techniques. There are many proposed and used estimators for the investigation of scaling properties in the financial and economic literature. In this section we start with the seminal work (Hurst 1951) on re-scaled range statistical analysis R/S with its complement (Hurst *et al.* 1965) which gives an estimator for the Hurst exponent. Indeed, the re-scaled range statistical analysis (R/S analysis) was first introduced by Hurst himself to describe the long-term dependence of water levels in rivers and reservoirs. It provides a sensitive method for revealing long-run correlations in random processes. This analysis can distinguish time series that are not correlated from correlated time series. What mainly makes the Hurst analysis appealing is that all this information about a complex signal is contained in one parameter only: the *Hurst exponent*.

Let us consider a time series $X(t)$ defined at discrete time intervals $t = \nu, 2\nu, 3\nu, \ldots, k\nu$. Let us define the average over a period T (which must be an entire multiple of ν) as

$$\langle X \rangle_T = \frac{\nu}{T} \sum_{k=1}^{T/\nu} X(k\nu). \tag{10}$$

The difference between the maximum and the minimum values of $X(t)$ in the interval $[\nu, T]$ is called the range R, which is given by

$$R(T) = \max[X(t)]_{\nu \le t \le T} - \min[X(t)]_{\nu \le t \le T}. \tag{11}$$

The Hurst exponent H is defined from the scaling property of the ratio

$$\frac{R(T)}{S(T)} \propto \left(\frac{T}{\nu}\right)^H, \tag{12}$$

where $S(T)$ is the standard deviation:

$$S(T) = \sqrt{\frac{\nu}{T} \sum_{k=1}^{T/\nu} [X(k\nu) - \langle X \rangle_T]^2}. \tag{13}$$

The Hurst exponent is sensitive to the long-range statistical dependence in the signal. It was proved by Hurst *et al.* (1965) and Feller (1971) that the asymptotic behaviour for any *independent random process* (Poisson process) with finite variance is given by

$$\frac{R(T)}{S(T)} = \left(\frac{\pi}{2\nu} T\right)^{1/2}, \tag{14}$$

which implies $H = 1/2$. However, many processes in nature are not independent random processes, but, on the contrary, show significant long-term correlations. In this case the asymptotic scaling law is modified and R/S is asymptotically given by the power law behaviour in equation 12 with $H \neq 0.5$.

It should be noted that the original Hurst R/S approach is very sensitive to the presence of short memory, heteroskedasticity, and multiple scale behaviour. Such a lack of robustness has been discussed in the literature (see, for instance, Lo (1991), Teverovsky *et al.* (1999), Weron and Przybylowsciz (2000) and Weron (2002)) and several alternative approaches have been proposed. Also, the fact that the range relies on maxima and minima makes the method error-prone because any outlier present in the data would have a strong influence on the range.

Lo (1991) suggested a modified version of the R/S analysis that can detect long-term memory in the presence of short-term dependence (Moody and Wu 1996). The modified R/S statistic differs from the classical R/S statistic only in its denominator, adding some weights and covariance estimators to the standard deviation (Newey and West 1987). In this modified R/S, a problem is choosing the truncation lag q. Andrews (1991) showed that when q becomes large relative to the sample size N, the finite-sample distribution of the estimator can be radically different from its asymptotic limit. However, the value chosen for q must not be too small, since the autocorrelation beyond lag q may be substantial and should be included in the weighted sum. The truncation lag must thus be chosen with some consideration. Despite these difficulties, several authors are still using this estimator, trying to avoid Lo's critique and proposing filtering procedures (Cajueiro and Tabak 2004, 2005).

5. Generalized Hurst exponent

The generalized Hurst exponent method is essentially a tool to study directly the scaling properties of the data via the qth-order moments of the distribution of the increments. The qth-order moments are much less sensitive to outliers than the maxima/minima and different exponents q are associated with different characterizations of the multi-scaling complexity of the signal. This type of analysis combines the sensitivity to any type of dependence in the data and a computationally straightforward and simple algorithm.

As shown in section 4, the Hurst analysis examines if some statistical properties of time series $X(t)$ (with $t = \nu, 2\nu, \ldots, k\nu, \ldots, T$) scale with the time-resolution (ν) and the observation period (T). Such a scaling is characterized by an exponent H which is commonly associated with the long-term statistical dependence of the signal. A generalization of the approach proposed by Hurst should therefore be associated with the scaling behaviour of statistically significant variables constructed from the time series. In this case, the qth-order moments of the distribution of the increments are used (Barabasi and Vicsek 1991, Mandelbrot 1997). This is a good quantity to characterize the statistical evolution of a stochastic variable $X(t)$. It is defined as

$$K_q(\tau) = \frac{\langle |X(t+\tau) - X(t)|^q \rangle}{\langle |X(t)|^q \rangle}, \tag{15}$$

where the time interval τ can vary between ν and $\tau_{\max}$. (Note that, for $q = 2$, $K_q(\tau)$ is proportional to the autocorrelation function: $a(\tau) = \langle X(t+\tau)X(t) \rangle$.)

The generalized Hurst exponent $H(q)$† can be defined from the scaling behaviour of $K_q(\tau)$ if it follows the relation (Groenendijk *et al.* 1998)

$$K_q(\tau) \sim \left(\frac{\tau}{\nu}\right)^{qH(q)}. \tag{16}$$

Within this framework, two kinds of process can be distinguished: (i) a process where $H(q) = H$ is constant and independent of q; and (ii) a process with $H(q)$ not constant. The first case is characteristic of uniscaling or unifractal processes and its scaling behaviour is determined from a unique constant H that coincides with the Hurst coefficient or the self-affine index, as already stated in section 3. This is indeed the case for self-affine processes where $qH(q)$ is linear ($H(q) = H$) and fully determined by its index H. In the second case, when $H(q)$ depends on q, the process is commonly called multi-scaling (or multi-fractal) (West 1985, Feder 1988) and different exponents characterize the scaling of different q-moments of the distribution. Therefore, the nonlinearity of the empirical function $qH(q)$ is a solid argument against the Brownian, fractional Brownian, Lévy, and fractional Lévy models, which are all additive models, therefore giving for $qH(q)$ straight lines or portions of straight lines.

For some values of q, the exponents are associated with special features. For instance, when $q = 1$, $H(1)$ describes the scaling behaviour of the absolute values of the increments. The value of this exponent is expected to be closely related to the original Hurst exponent, H, which is indeed associated with the scaling of the absolute spread in the increments. The exponent at $q = 2$ is associated with the scaling of the autocorrelation function and is related to the power spectrum (Flandrin 1989). A special case is

†We use H without parentheses as the original Hurst exponent, and $H(q)$ as the generalized Hurst exponent.

associated with the value of $q = q^*$, at which $q^* H(q^*) = 1$. At this value of q, the moment $K_{q^*}(\tau)$ scales linearly in τ (Mandelbrot 1997). Since $qH(q)$ is, *in general*, a monotonic growing function of q, all the moments $H_q(\tau)$ with $q < q^*$ will scale slower than τ, whereas all the moments with $q > q^*$ will scale faster than τ. The point q^* is therefore a threshold value. Clearly, in the unifractal case, $H(1) = H(2) = H(q^*)$. All these quantities will be equal to 1/2 for Brownian motion and they would be equal to $H \neq 0.5$ for fractional Brownian motion. However, for more complex processes, these coefficients do not, in general, coincide. In section 8.1 I will report empirical results for $H(q)$ when $q = 1$ and 2.

6. Scaling exponents in the frequency domain

For financial time series, as well as for many other stochastic processes, the spectral density $S(f)$ is empirically found to scale with the frequency f as a power law: $S(f) \propto f^{-\beta}$. It is easy to show using a simple argument that this scaling in the frequency domain should be related to the scaling in the time domain. Indeed, it is known that the spectrum $S(f)$ of the signal $X(t)$ can be conveniently calculated from the Fourier transform of the autocorrelation function (Wiener–Khinchin theorem). On the other hand, the autocorrelation function of $X(t)$ is proportional to the second moment of the distribution of the increments, which, from equation (16), is assumed to scale as $K_2 \sim \tau^{2H(2)}$. But, the components of the Fourier transform of a function that behaves in the time domain as τ^{α} are proportional to $f^{-\alpha-1}$ in the frequency domain. Therefore, the power spectrum of a signal that scales as equation (16) must behave as

$$S(f) \propto f^{-2H(2)-1}. \tag{17}$$

Consequently, the slope β of the power spectrum is related to the generalized Hurst exponent for $q = 2$ via $\beta = 1 + 2H(2)$. Note that equation (17) is obtained only by assuming that the signal $X(t)$ has scaling behaviour in accordance with equation (16) without making any hypothesis with respect to the kind of underlying mechanism that might lead to such scaling behaviour.

7. Other methods

There has been a proliferation of papers proposing different techniques and providing comparison studies between them (Taqqu *et al.* 1995). Let us start by mentioning the most popular: the detrended fluctuation analysis (DFA) (Peng *et al.* 1994, Vandewalle and Ausloos 1997, Viswanathan *et al.* 1997, Ausloos *et al.* 1999, Janosi *et al.* 1999, Liu *et al.* 1999, Raberto *et al.* 1999, Stanley *et al.* 1999, Ausloos 2000, Hu *et al.* 2001, Chen *et al.* 2002, Ivanova and Ausloos 2002, Costa and Vasconcelos 2003) and its generalization (Kantelhardt *et al.* 2002, Matia *et al.* 2003, Oswiecimka *et al.* 2005); the moving-average analysis technique (Ellinger 1971) and its comparison with the DFA (Alessio *et al.* 2002, Carbone *et al.* 2004); the periodogram regression (GPH method) (Geweke and Porter-Hudak 1983); the (m, k)-Zipf method (Zipf 1949); the Average Wavelet Coefficient Method (Mehrabi *et al.* 1997, Simonsen *et al.* 1998, Percival and Walden 2000, Gençay *et al.* 2001); and the ARFIMA estimation by exact maximum likelihood (ML) (Sowell 1992, Ellis 1999, Grau-Carles 2000). Let us stress that no method exists whose performance has no deficiencies. The use of each of the above estimators can be accompanied by both advantages and disadvantages. For instance, simple traditional estimators can be seriously biased. On the other hand, asymptotically unbiased estimators derived from Gaussian ML estimation are available, but these are parametric methods which require a parameterized family of model processes to be chosen *a priori*, and which cannot be implemented exactly in practice for large data sets due to the high computational complexity and memory requirements (Phillips 1999a,b, Phillips and Shimotsu 2001). Analytic approximations have been suggested (Whittle estimator) but, in most cases (Beran 1994), computational difficulties remain, motivating a further approximation: the discretization of the frequency-domain integration. Even with all these approximations the Whittle estimator still has a significantly high overall computational cost, and problems of convergence to local minima rather than to the absolute minimum may also be encountered. In this framework, connections to multi-scaling/multi-affine analysis (the q-order height–height correlation) have been made in various papers (Vandewalle and Ausloos 1998a,b, Ivanova and Ausloos 1999, Selçuk 2004). Di Matteo *et al.* (2003, 2005) studied the scaling properties of different financial data using a different and alternative method: the generalized Hurst exponent method $H(q)$ described in section 5.

8. Empirical investigation

8.1. *Generalized Hurst exponent H(q) for q = 1 and q = 2*

In this section we report some empirical results from the most comprehensive and extended empirical analysis of the scaling properties of several financial markets ever performed. The analysis includes: 32 Stock market indices, 29 Foreign exchange rates and 28 fixed income instruments at different stages of development (mature and liquid markets, emerging and less liquid markets) (Di Matteo *et al.* 2003, 2005). As an example, the behaviour of JPY/USD and the Nikkei 225 (Japan) and THB/USD and the Bangkok SET (Thailand) as a function of time t are shown in figure 1 for the period 1997–2001. Another example is given in figure 2, which shows the Treasury rates time series as a function of t at different maturity dates in the period 1997–2001 and the Eurodollar time series as a function of t at different

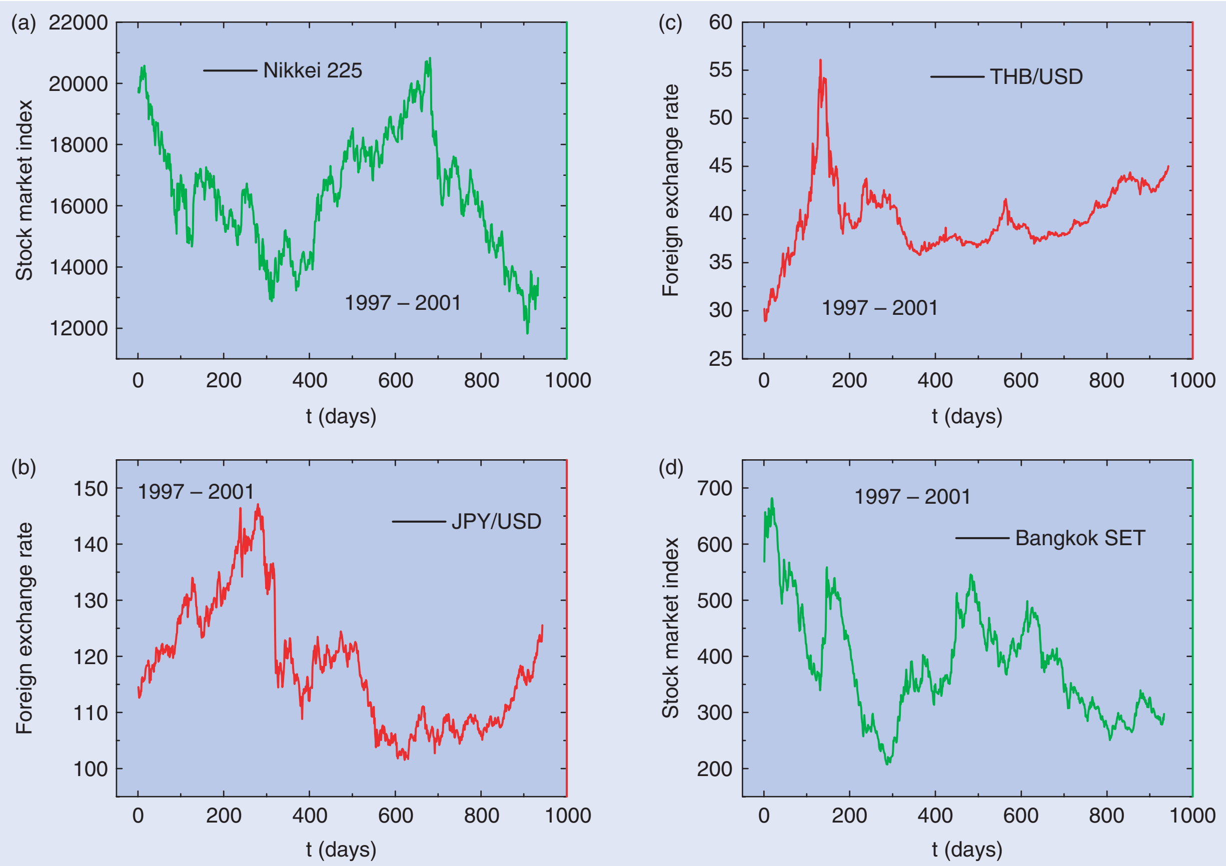

Figure 1. The Foreign Exchange rates and the Stock Market indices as a function of time t in the period 1997–2001: (*a*) Nikkei 225; (*b*) JPY/USD; (*c*) THB/USD; (*d*) Bangkok SET.

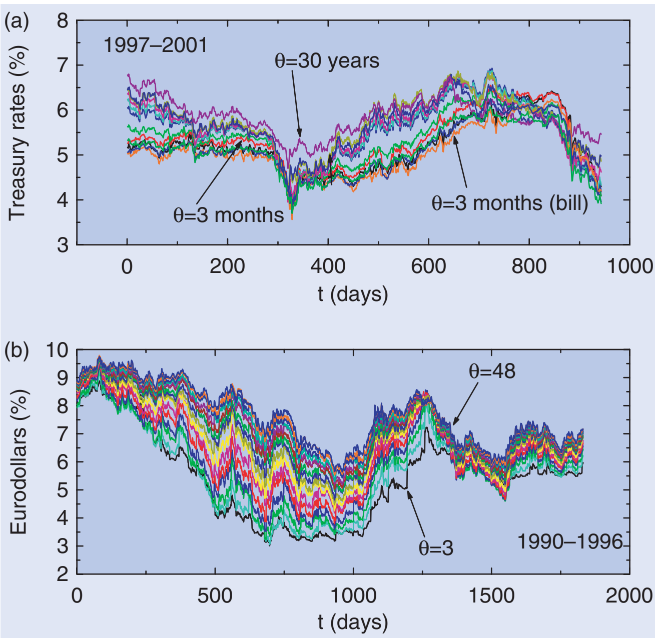

Figure 2. (*a*) The Treasury rates at 'constant maturity' as a function of t in the period 1997–2001. Each curve corresponds to a maturity date θ, ranging from 3 months to 30 years, and Treasury bill rates to a maturity date $\theta = 3$ and 6 months and 1 year. (*b*) Eurodollar interest rates as a function of t in the period 1990–1996. Each curve corresponds to a maturity date θ, ranging from 3 to 48 months.

maturity dates in the period 1990–1996 (Di Matteo and Aste 2002, Di Matteo *et al*. 2004). This empirical analysis is performed on the daily time series, which typically span periods between 1000 and 3000 days. In particular, the analysis is performed on the time series themselves for the Treasury rates and Eurodollar, whereas the returns from the logarithmic price $X(t) = \ln(P(t))$ are used for Foreign exchange and Stock market indices. The q-order moments $K_q(\tau)$ (defined in equation (15)), with τ in the range between $\nu = 1$ day and $\tau_{\max}$ days, are computed and the scaling behaviour, given by equation (16), is verified to be followed. For instance, in figures 3 and 4 the scaling behaviour of $K_q(\tau)$, in agreement with equation (16), is shown in the period from 1990 to 2001 for the Nikkei 225 (figure 3) and the Bangkok Set (figure 4). Each curve corresponds to different fixed values of q ranging from $q = 1$ to $q = 3$, whereas τ varies from 1 to 19 days. Moreover, this scaling behaviour has been carefully checked to hold for all the financial time series studied in Di Matteo *et al*. (2005). All these time series exhibit evidence of multi-scaling behaviour, showing curves of $qH(q)$ as a function of q not linear in q, but slightly bending below the linear trend (see figure 5). This is a sign of deviation from the Brownian, fractional Brownian, Lévy and fractional Lévy models, as already seen in Foreign exchange rates (Müller *et al*. 1990). (Other cases showing marked deviations from Brownian motion have been discussed elsewhere

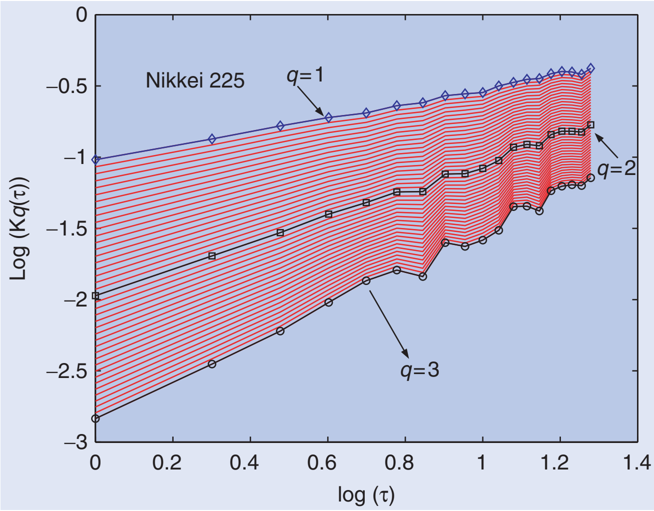

Figure 3. $K_q(\tau)$ as a function of τ on a log–log scale for the Nikkei 225 time series in the period from 1990 to 2001 (τ varies from 1 to 19 days). Each curve corresponds to different fixed values of q ranging from $q=1$ to $q=3$. Clearly shown are the curves corresponding to $q=1(\Diamond)$, $q=2(\Box)$ and $q=3(\circ)$.

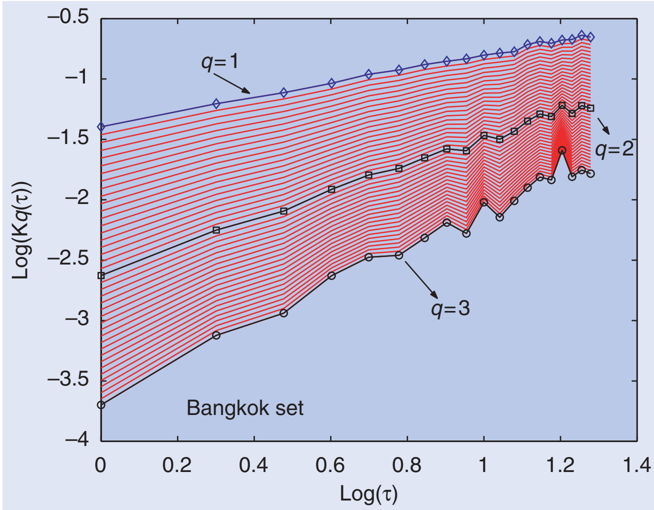

Figure 4. $K_q(\tau)$ as a function of τ on a log–log scale for the Bangkok Set time series in the period from 1990 to 2001 (τ varies from 1 to 19 days). Each curve corresponds to different fixed values of q ranging from $q=1$ to $q=3$. Clearly shown are curves corresponding to $q=1(\Diamond)$, $q=2(\Box)$ and $q=3(\circ)$.

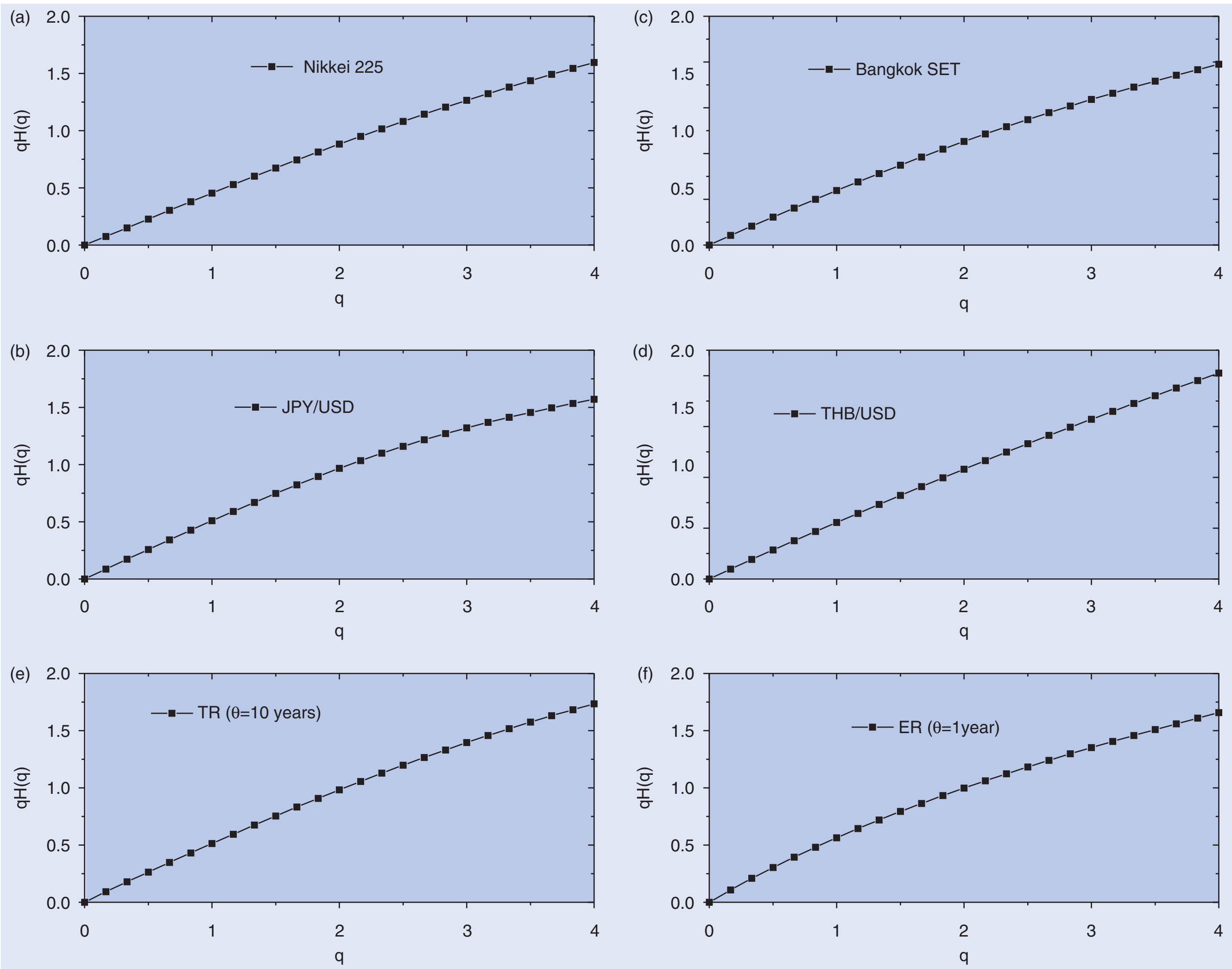

Figure 5. The function $qH(q)$ vs. q in the period from 1997 to 2001. (*a*) Japan (Nikkei 225); (*b*) Japan (JPY/USD); (*c*) Thailand (Bangkok SET); (*d*) Thailand (THB/USD); (*e*) Treasury rates having maturity dates of $\theta=10$ years; (*f*) Eurodollar rates having a maturity date of $\theta=1$ year. For (*f*) the period is 1990–1996.

(Vandewalle and Ausloos 1998a,c, Ivanova and Ausloos 1999, Ausloos and Ivanova 2001a,b).)

The results for the scaling exponents $H(q)$ computed for $q=1$ and $q=2$, for all assets (Di Matteo *et al.* 2003, 2005) and different markets, are reported in figures 6 and 7, respectively. It emerges that, for fixed income instruments (figures 6(*a*) and 7(*a*)), $H(2)$ is close to 0.5, while $H(1)$ is rather systematically above 0.5 (with the 3 month Eurodollar rate showing a more pronounced deviation because it is directly influenced by the actions of central banks). On the other hand, as far as Stock markets are concerned, the generalized Hurst exponents $H(1)$ and $H(2)$ show remarkable differences between developed and emerging markets. In particular, the values of $H(1)$, plotted in figure 6(*b*), present a differentiation across 0.5 with high values of $H(1)$ associated with emerging markets and low values of $H(1)$ associated with developed markets. Figure 6(*b*) shows the ordering of the stock markets from left to right in ascending order of $H(1)$. One can see that such an ordering corresponds to the order one would intuitively give in terms of maturity of the markets. Moreover, the different assets can be classified into three different categories (see figure 7(*b*)).

1. First, those that have an exponent $H(2)>0.5$, which includes all indices of emerging markets and the BCI 30 (Italy), IBEX 35 (Spain) and the Hang Seng (Hong Kong).
2. The second category includes the data exhibiting $H(2)\sim 0.5$ (within error bars). This category includes: the FTSE 100 (UK), AEX (Netherlands), DAX (Germany), Swiss Market (Switzerland), Top 30 Capital (New Zealand), Tel Aviv 25 (Israel), Seoul Composite (South Korea) and Toronto SE 100 (Canada).
3. The third category is associated with $H(2)<0.5$ and includes the following data: the Nasdaq 100 (US), S&P 500 (US), Nikkei 225 (Japan), Dow Jones Industrial Average (US), CAC 40 (France) and All Ordinaries (Australia).

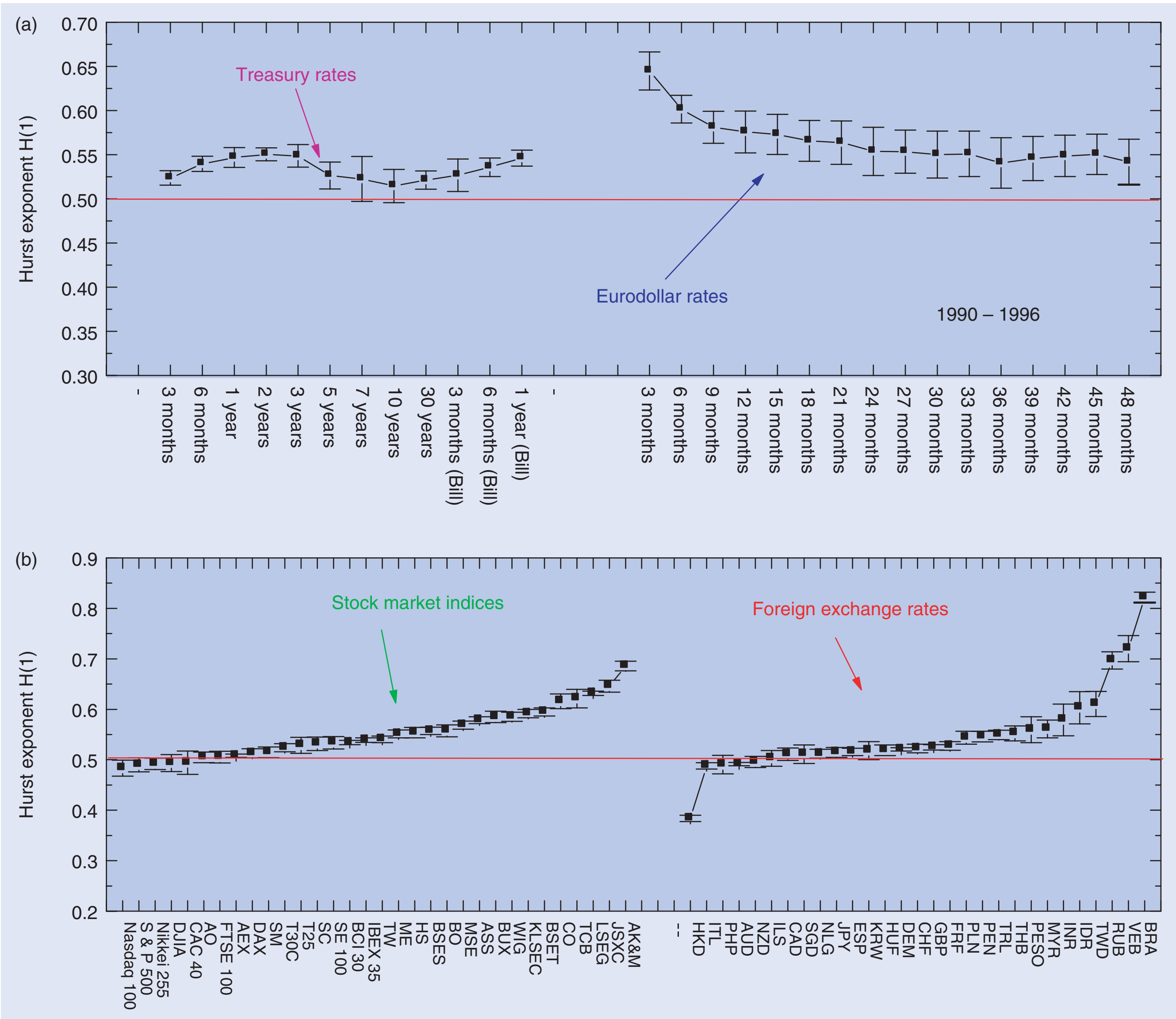

Figure 6. The Hurst exponent $H(1)$ for the Treasury and Eurodollar rate time series (*a*) and for the Stock Market indices and Foreign Exchange rates (*b*). On the x axis we report the corresponding maturity dates (*a*) and the corresponding data sets (*b*).

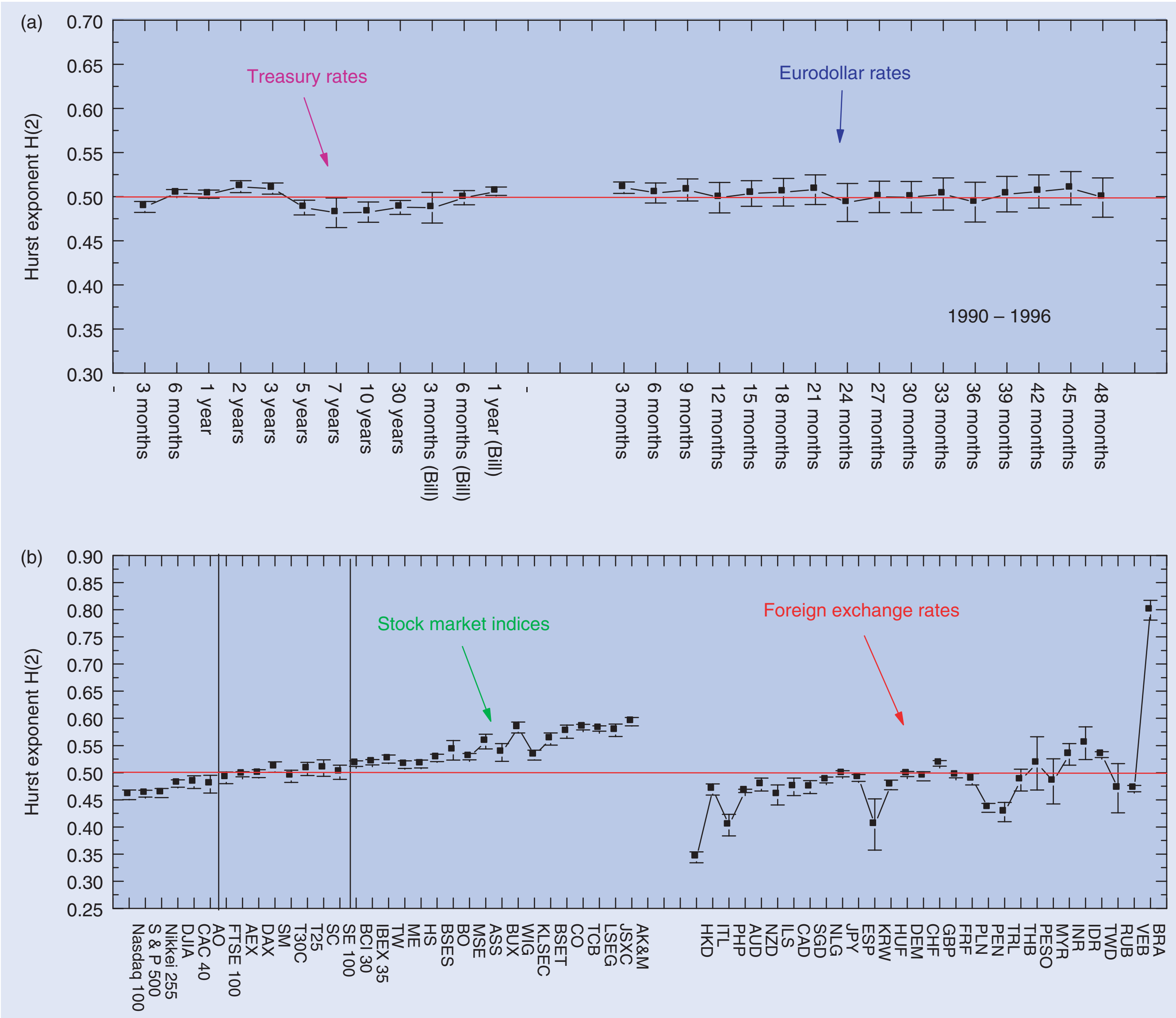

Figure 7. The Hurst exponent $H(2)$ for the Treasury and Eurodollar rate time series (a) and for the Stock Market indices and Foreign Exchange rates (b). On the x axis we report the corresponding maturity dates (a) and the corresponding data sets (b).

Therefore, all emerging markets exhibit $H(2) \geq 0.5$, whereas all well-developed markets show $H(2) \leq 0.5$. This simple classification cannot be achieved by other means. For instance, the use of the Sharpe Ratio (Sharpe 1994) does not achieve such a clear-cut categorization. This ratio requires a benchmark risk-free return that is not always available for emerging markets.

The Foreign Exchange rates exhibit $H(1) > 0.5$ quite systematically. This is consistent with previous results computed with high-frequency data (Müller *et al.* 1990), although the values here are slightly lower. An exception with pronounced $H(1) < 0.5$ is HKD/USD (Hong Kong) (figure 6(b)). This FX rate is, or has been, at one point pegged to the USD, which is why its exponent differs from the others. In the class $H(1) \sim 0.5$ we have: ITL/USD (Italy), PHP/USD (Philippines), AUD/USD (Australia), NZD/USD (New Zealand), ILS/USD (Israel), CAD/USD (Canada), SGD/USD (Singapore), NLG/USD (Netherlands) and JPY/USD (Japan). On the other hand, the values of $H(2)$ (figure 7(b)) show a much greater tendency to be <0.5 with some strong deviations, such as: HKD/USD (Hong Kong), PHP/USD (Philippines), KRW/USD (South Korea), PEN/USD (Peru) and TRL/USD (Turkey). Values $H(2) > 0.5$ are found for: GBP/USD (United Kingdom), PESO/USD (Mexico), INR/USD (India), IDR/USD (Indonesia), TWD/USD (Taiwan) and BRA/USD (Brazil).

These analyses were also performed over different time periods (Di Matteo *et al.* 2003, 2005). In particular, the results for sub-periods of 250 days indicate that there are significant changes in market behaviour over different time periods. This phenomenon was also detected by Dacorogna *et al.* (2001a) when studying Exchange rates that were part of the European

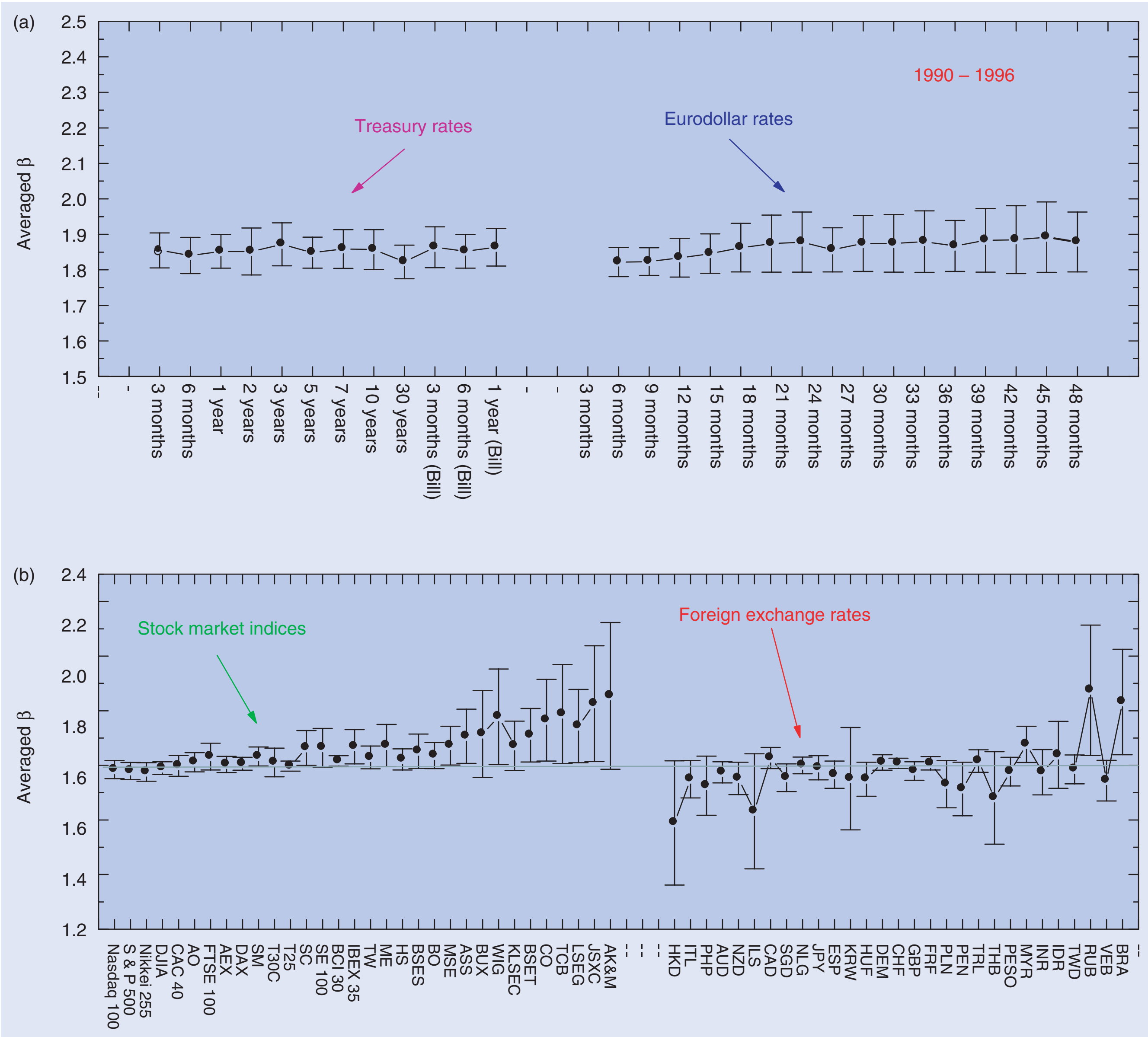

Figure 8. (*a*) The averaged β values computed from the power spectra (mean square regression) of the Treasury and Eurodollar rate time series (the corresponding maturity dates are reported on the x axis). (*b*) The averaged β values computed from the power spectra of the Stock Market indices and Foreign Exchange rates. The horizontal line corresponds to the value of β obtained from the simulated random walks (the corresponding data sets are reported on the x axis).

Monetary System. It seems that $H(1)$ is particularly sensitive to institutional changes in the market. The scaling exponents cannot be assumed to be constant over time if a market is experiencing major institutional changes. Nevertheless, well-developed markets have values of $H(2)$ that are, on average, smaller than the emerging values and the weakest markets have oscillation bands that stay above 0.5, whereas the strongest markets have oscillation bands that contain 0.5. Temporal variability is a sign that the exponents are sensitive to institutional changes in the market, reinforcing the idea of using them as indicators of the maturity of the market.

Furthermore, the robustness and reliability of the generalized Hurst exponent method has also been tested extensively in several ways: first, by comparing theoretical exponents with the results of Monte Carlo simulations using three distinct random generators (Marsaglia and Zaman 1994); second, by varying the maximum time step (τ_{max}) in the analysis; third, by applying the Jackknife method (Kunsch 1989) to produce several samples; fourth, by varying the time-window size to analyse the temporal stability; and fifth, by computing results for detrended and non-detrended time series.

8.2. Spectral analysis

In order to investigate empirically the statistical properties of the time series in the frequency domain a spectral analysis has been performed (Di Matteo *et al.* 2005). The power spectral density (PSD) (Kay and Marple 1981) was computed using the periodogram

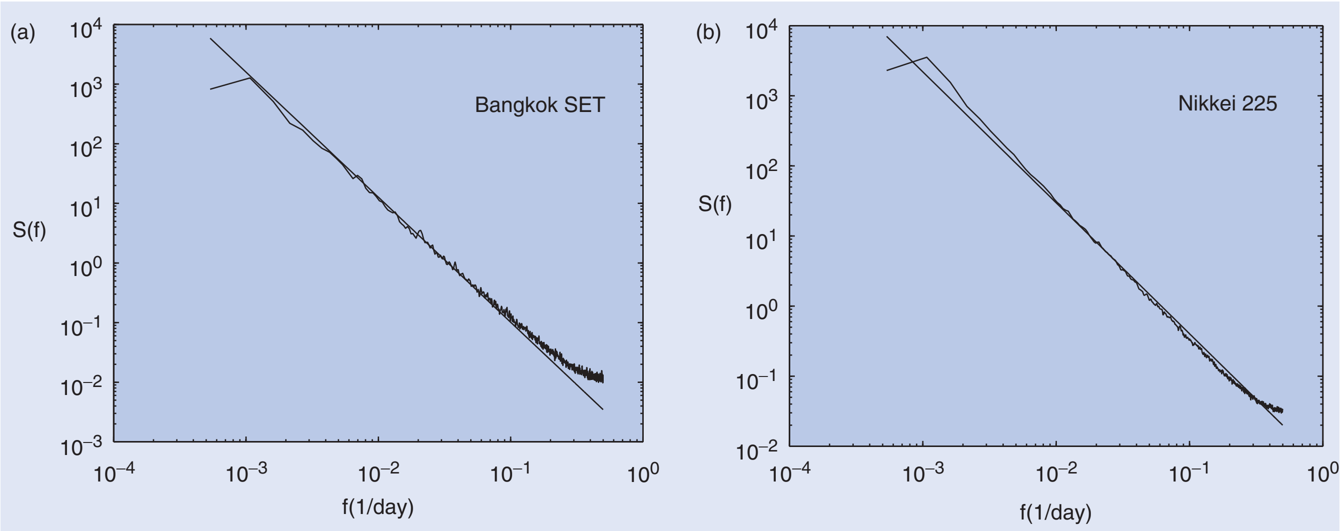

Figure 9. The power spectra of the Stock Market indices compared with the behaviour of $f^{-2H(2)-1}$ (straight lines on a log–log scale) computed using the Hurst exponent values in the period 1997–2001. (*a*) Thailand (Bangkok SET) and (*b*) Japan (Nikkei 225). The line is the prediction from the generalized Hurst exponent $H(2)$ (equation (17)).

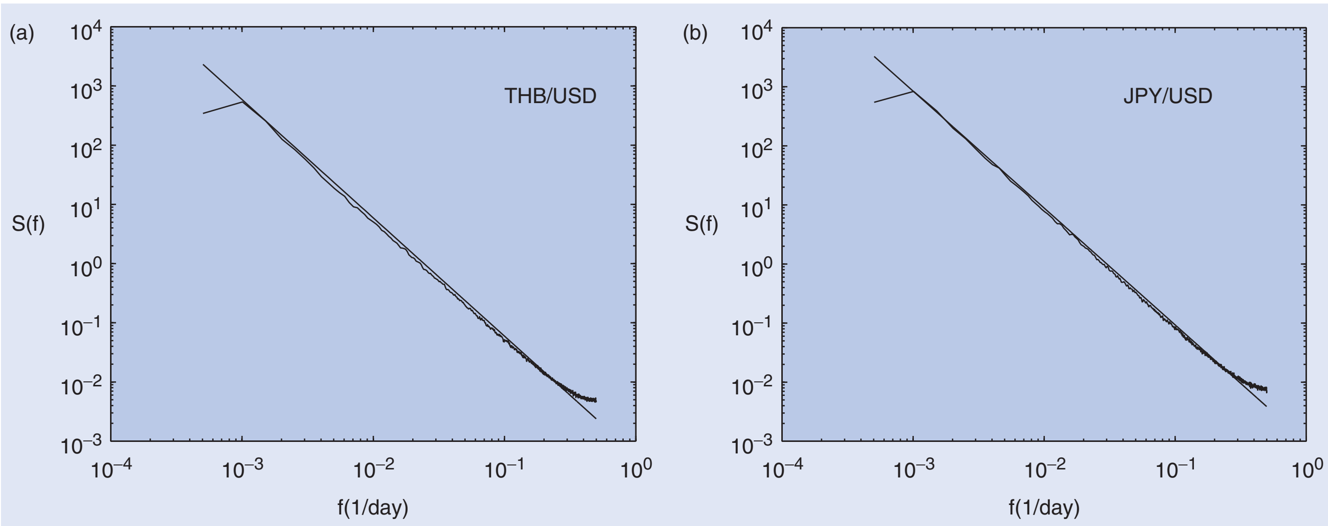

Figure 10. The power spectra of the Foreign Exchange rates compared with the behaviour of $f^{-2H(2)-1}$ (straight lines on a log–log scale) computed using the Hurst exponent values in the period 1997–2001. (*a*) Thailand (THB/USD) and (*b*) Japan (JPY/USD). The line is the prediction from the generalized Hurst exponent $H(2)$ (equation (17)).

approach (which is currently one of the most popular and computationally efficient PSD estimators). This is a sensitive way of estimating the limits of the scaling regime of data increments.

The results for some Stock market indices, Foreign exchange rates, Treasury rates and Eurodollar data in the periods 1997–2001 and 1990–1996 are shown in figures 9, 10 and 11. As can be seen the power spectra show clear power law behaviour: $S(f) \sim f^{-\beta}$. This behaviour holds for all other data (Di Matteo *et al*. 2005).

The power spectra coefficients β have been calculated via a mean square regression on a log–log scale for the Treasury rates and Eurodollar rates (figure 8(*a*)) and for the Stock Market indices and Foreign Exchange rates (figure 8(*b*)). The values reported in figure 8 are the average of β evaluated over different windows and the error bars are their standard deviations. These values differ from the spectral density exponent expected for pure Brownian motion ($\beta = 2$) (Feller 1971). However, Di Matteo *et al*. (2005) showed that this method is biased: power spectra exponents around 1.8 were found for random walks from three different random number generators. It should also be noted that the power spectrum is only a second-order statistic and its slope is not sufficient to validate a particular scaling model: it gives only partial information concerning the statistics of the process.

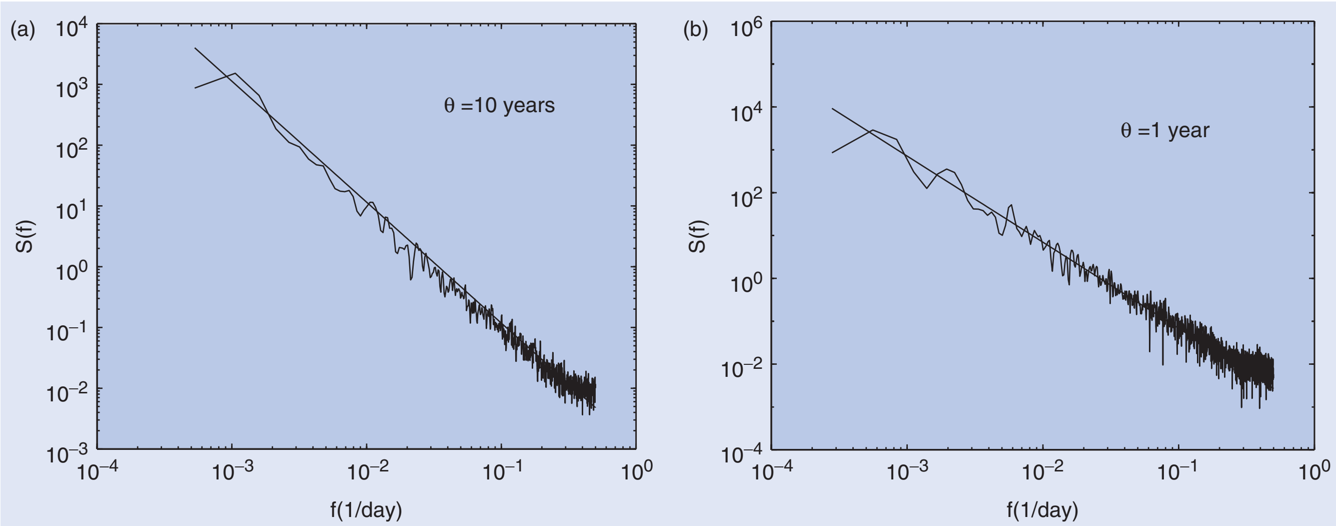

Figure 11. The power spectra compared with the behaviour of $f^{-2H(2)-1}$ (straight lines on a log–log scale) computed using the Hurst exponent values in the period 1997–2001. (*a*) Treasury rates having maturity dates of $\theta = 10$ years; (*b*) Eurodollar rates having a maturity date of $\theta = 1$ year in the period 1990–1996. The line is the prediction from the generalized Hurst exponent $H(2)$ (equation (17)).

8.3. Scaling spectral density and Hurst exponent

In this section the behaviour of the power spectra $S(f)$ is compared with the function $f^{-2H(2)-1}$, which, according to equation (17), is the scaling behaviour expected in the frequency domain for a time series that scales in time with a generalized Hurst exponent $H(2)$. The results of the comparison for Foreign Exchange rates and Stock Market indices for Thailand and Japan (in the period 1997–2001) are reported in figures 9 and 10 and those for the Treasury and the Eurodollar rates having maturity dates $\theta = 10$ years and $\theta = 1$ year (in the period 1990–1996) are reported in figure 11. As can be seen the agreement between the power spectra behaviour and the prediction from the generalized Hurst analysis is very satisfactory. This result also holds for all the other data. Note that the values of $2H(2)+1$ do not, in general, coincide with the values of the power spectral exponents evaluated by means of the mean square regression. The method using the generalized Hurst exponent appears to be *more* powerful in catching the scaling behaviour, even in the frequency domain.

9. Conclusion

In the literature, scaling analysis has been widely applied and used in different contexts. In this review, we have discussed the different tools used for estimating the scaling exponents, stressing their advantages and disadvantages.

Particular emphasis has been given to the generalized Hurst exponent approach, a suitable tool for describing the multi-scaling properties in financial time series. We have shown that this approach provides a natural, unbiased, statistically and computationally efficient estimator able to capture very well the scaling features of financial fluctuations. This method is powerful and robust and is not biased, as other methods are. On the other hand, estimations of the scaling exponents in the frequency domain show that this method is affected by a certain bias. Finally, the method using the generalized Hurst exponent describes well the scaling behaviour, even in the frequency domain.

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