European Options Pricing Using Monte Carlo Simulation

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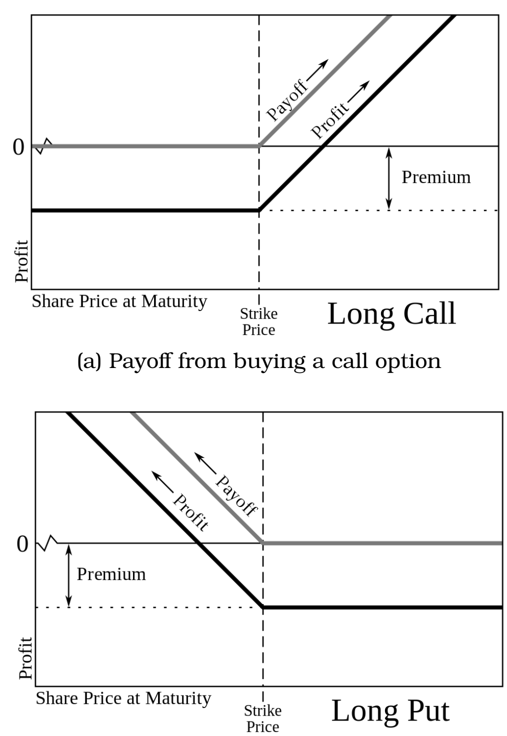
European options can be priced using the analytical solution of the Black-Scholes-Merton differential equation with the appropriate boundary conditions. A different approach and the one commonly used in situations where no analytical solution is available is the Monte Carlo Simulation. We present the results of Monte Carlo simulations for pricing European options and we compare with the analytical solution from the Black-Scholes Merton model. In addition, we examine the effects of the different parameters of a Monte Carlo simulation.

I. Introduction

I.1. Options

An option is a type of security which gives the owner the right buy (call option) or sell (put option) the underlying asset at a predefined strike price. The one who issues an option, called the writer, must deliver to the buyer a specified number of shares if the latter decides to exercise the option. The buyer pays the writer a premium in exchange for writing the option.

The European option can only be exercised at maturity which makes it a very simple kind of derivative. European options can be priced using the analytical solution of the Black-Scholes-Merton differential equation. However, MC is another way to acquire the price of a European option. The analytical result can be used as a benchmark for the simulations.



(a) Payoff from buying a call option

(b) Payoff from buying a put option

Figure 1: Payoff and profit of a call and a put european option with respect to the price of the underlying asset at maturity

Figure 1 presents the payoff and the profit of the two types of the european option relative to the price of the underlying asset at maturity. A call option can be exercised if the asset price is greater than the strike price of the option. Otherwise, the asset can be acquired in the market at a lower price. Even in the case where the asset price is greater than the strike price, the option may not yield positive profit because of the premium paid in order to buy the option. A put option, shown in figure 1b, exhibits the opposite behavior from a call option. Giving the owner the right to sell the asset, this option yields higher profit while the market price is less than the strike price.

Price at Maturity	Stock Return	Option Return
95	-5%	-100%
100	0%	-100%
105	5%	-100%
110	10%	0%
115	15%	100%

Table 1: Comparison of the return of a call option and the return of the underlying stock. The initial price is considered to be \$100, the strike price is \$105 and the premium is \$5.

Table 1 intends to show the high risk of dealing with options. Options are favored by high volatility because they yield great profits only when the price of the stock at maturity is significantly different from the initial price. However, a movement of the price in the wrong direction could cause losses of 100% of the initial investment.

The risky nature of the options makes them suitable for two situations. First, the correct prediction of the magnitude and the timing of the movement of the underlying asset can yield tremendous profits. Second, options can be used for hedging risk from other risky investments. For example, a put option could provide insurance to the buyer against a drop in the price of the stock.

I.2. The Monte Carlo Approach

Monte Carlo (MC) simulation is a robust and widely used method to price derivatives and estimate the risk of a portfolio [1]. The stochastic nature of the MC simulation makes it a suitable technique for problems with many sources of uncertainty. Hence, MC is essential for pricing exotic options like lookback and asian options or options that are dependent on a basket of underlying assets rather than just a single asset.

A MC simulation repeats a process many times attempting to predict all the possible future outcomes. At the end of the simulation, a number of random trials produce a distribution of outcomes that can be analyzed. In the case of option pricing, the outcomes are the future price of the stock.

A model of the behavior of the stock price needs to be specified in order to be used with the MC simulation. One of the most common models in finance is the geometric Brownian motion (GBM). In general, a quantity B is said to follow a Wiener process if

- $B(0) = x$
- $B(t_n) - B(t_{n-1})$, $B(t_{n-1}) - B(t_{n-2})$, ..., $B(t_1) - B(0)$ are independent random variables
- For all $t \geq 0$ and $\Delta t > 0$, $B(t+\Delta t) - B(t)$ are normally distributed with expectation 0 and standard deviation $\sqrt{\Delta t}$

If W_t is a Wiener process, then a stochastic process S_t is said to follow a GBM if it satisfies the stochastic differential equation

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t \tag{1}$$

where μ is the drift used to model deterministic trends and σ is the volatility used to model unpredictable events. The stochastic process S_t represents the price of the stock at any given time t while μ and σ is the expected annual return and annual volatility of the stock respectively.

In case the variable x follows the Ito process

$$\mathrm{d}x = a(x,t)\,\mathrm{d}t + b(x,t)\,\mathrm{d}z \tag{2}$$

then a function $G(x,t)$ follows the process

$$\mathrm{d}G = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\right)\mathrm{d}t + \frac{\partial G}{\partial x}b\,\mathrm{d}W \tag{3}$$

Hence, using $\mathrm{d}S = \mu S\,\mathrm{d}t + \sigma S\,\mathrm{d}W$, equation (3) yields

$$\begin{aligned}\mathrm{d}G = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)\mathrm{d}t \\ + \frac{\partial G}{\partial S}\sigma S\,\mathrm{d}W\end{aligned} \tag{4}$$

Using equation (4) and $G = \ln S$, it can be readily shown that the change of $\ln S$ between time $t = 0$ and a future time $t = T$ is normally distributed with mean $(\mu - \sigma^2/2)T$ and variance $\sigma^2 T$ as described in equation (5)

$$\ln S_T - \ln S_0 \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right] \tag{5}$$

where ϕ denotes the normal distribution.

Since $\ln S_t$ follows a normal distribution, the stock price S_t follows a lognormal distribution with expectation value $E(S_t) = S_0 e^{\mu T}$. The expected return μ is driven by the riskiness of the stock and the interest rates in the economy. Higher risk causes higher expected returns by the investors. Also, the higher the level of interest rates in the economy the higher the expected return required on any given stock.

I.3. The Black-Scholes-Merton Model

In the early 1970s, Black, Scholes and Merton developed what has become known as the Black-Scholes-Merton (BSM) model [2, 3]. This model has had a huge influence on the pricing of options. In order to use the model, certain assumptions must be made. The assumptions are as follows

1. The stock price follows the process given by equation (1)
2. The short selling of securities with full use of proceeds is permitted
3. There are no fees and taxes. All securities are perfectly divisible
4. There are no dividends
5. There is no arbitrage
6. Trading is continuous
7. The risk free rate, r, is constant and the same for all maturities

The BSM differential equation can be derived using Ito's lemma and a riskless portfolio. Suppose that f is the price of a derivative of S. As a result, the variable f must be some function of S and t. Hence from Ito's Lemma we get,

$$\mathrm{d}f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\mathrm{d}t + \frac{\partial f}{\partial S}\sigma S\,\mathrm{d}W \tag{6}$$

Consider the portfolio which is short by one derivative and long by $\dfrac{\partial f}{\partial S}$ shares given by equation (7).

$$\Pi = -f + \frac{\partial f}{\partial S}S \tag{7}$$

The change $\Delta\Pi$ in the value of the portfolio in the time interval Δt is given by

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S \tag{8}$$

Using equations (1) and (6) into equation (8) yields

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t \tag{9}$$

This is a riskless portfolio because there is no dependence on any stochastic variable. For a risk-free portfolio the relationship $\Delta\Pi = r\Pi\Delta t$ holds. Hence, substituting in equation (9) the Black-Scholes-Merton differential equation can be written as

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf \tag{10}$$

This equation can be solved analytically for some certain cases. The appropriate boundary conditions for a European option are $f_{call} = \max(S - K, 0)$ and $f_{put} = \max(K - S, 0)$ where S is the stock price and K is the strike price. Then, for a non-dividend-paying stock, the prices of the call and the put options at time $t = 0$ are

$$c = S_0 N(d_1) - Ke^{-rT}N(d_2) \tag{11}$$

and

$$p = -S_0 N(-d_1) + Ke^{-rT}N(-d_2) \tag{12}$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

$N(x)$ is the cumulative probability distribution function for a standardized normal distribution $\phi(0, 1)$.

The variables of equation (10) are all independent of the risk preferences of the investors. In other words, the expected return μ is eliminated. This is the basis of risk neutral valuation. In a world where investors are risk neutral, the expected return on all investement assets is the risk-free rate of interest, r.

A derivative that provides a payoff at one particular time can be valued using risk-neutral valuation by assuming that the expected return from the underlying asset is the risk-free rate (i.e., assume $\mu = r$). The calculated expected payoff from the derivative should then be discounted at the risk-free interest rate.

II. Simulation Methodology

The stock price is assumed to follow geometric Brownian motion. Based on equation (1) and for a finite time step Δt, the Wiener process is $\epsilon\sqrt{\Delta t}$ where ϵ is a number sampled from a standardized normal distribution. Hence,

$$S_t = S_{t-1}\left(1 + \mu\,\Delta t + \sigma\epsilon\sqrt{\Delta t}\right) \tag{13}$$

The stock price follows a series of steps, where each step consists of a drift by the expected return μ and an upward or downward movement based on the volatility scaled by the random number ϵ. The expected return is considered to be the risk-free rate in order to price the option using risk neutral valuation.

The MC simulation consists of three main steps. First, using the GBM defined by equation (13), we calculate the stock price at maturity. Second, we calculate the payoff of the option based on the stock price and finally we discount the payoff at the risk-free rate to today's price. Repeating the above procedure for a reasonable number of times, gives a good estimate of the average payoff and the price of the option.

The simulations were performed using eight different input parameters. Those parameters and their default values are given in table 2. Unless otherwise stated, these values were used in all the simulations.

The expected annual return rate and the risk-free annual rate should be the same in order to get the correct estimation of the option. The simulated results are compared and normalized to the values given by equations (11) and (12).

Input Parameter	Default Value
Initial price	100
Strike price	102
Expected annual return	1%
Risk-free annual rate	1%
Annual volatility	20%
Number of steps	252
Years to maturity	1
Number of trials	2500

Table 2: Default input parameters for the MC simulation

III. Results

A typical output of random paths is presented in figure 2. The simulated stock prices at maturity produce a lognormal distribution as predicted by equation (5). Figure 3 shows the lognormal distribution of the stock prices at maturity for 250000 trials. This is a very computationally demanding calculation. In order to achieve optimal efficiency, the convergence of the simulations should be examined.

Convergence should be verified in all computational approaches. In the case of a MC simulation, the parameters that affect the convergence are the number of time steps and the number of trials. A set of significant time steps are $\{4, 52, 252, 500\}$. Using 1 year to maturity, 4 time steps represent one movement per quarter, 52 time steps represent weekly movements and 252 time steps are used to account for the 252 trading days per year. An additional value of 500 time steps roughly accounts for two movements per day.

First, the expected price of the stock for the given GBM is given by $E(S) = S_0 e^{rT}$. Thus, we examine the effects of the number of trials for different time steps. Figure 4 shows that even for the case of only 4 time steps, the average value of the simulations converges to the expected value as the number of trials increases. Figure 5 presents the case of the 252 time steps for more trials. This figure also supports that good convergence is achieved for more than 2500 trials. We also notice that the fluctuations of the simulated price versus the BSM price are minute above 2500 trials.

However, performing the same calculation for the convergence of the call option price, reveals that 4 time steps are insufficient to correctly valuate the option. In fact, the result is underestimated as shown in figure 6. The underestimation of the option price for few time steps can be readily seen in figure 7. An important observation from this figure is that once convergence is achieved, the number of time steps becomes irrelevant to the result of the option price.

Another significant parameter is the strike price K at maturity. In principle there are no forbidden values for the strike price. However, the strike price is closely related to the initial price, the expected return and the volatility. For example, if the initial price is $\$100$, the expected annual return is the risk-free rate 1% and the annual volatility is 20%, a strike price of $\$102$ for a call option is perfectly justified. On the other hand, a strike price of $\$150$ is unreasonable because it is extremely unlikely for the stock price to rise so much for the given expected return and volatility.

Figure 8 shows the convergence of the call price versus the time steps for different strike prices. It can be seen that higher strike prices not only cause slower convergence, but they also result in higher fluctuations. Even though the convergence is influenced, the change is minute compare to the increase in the uncertainty of the simulated price. The uncertainty can be viewed by the fluctuations of the normalized option price.

A way to examine this effect is presented in figure 9. In this case we compare the normalized call price for different strike prices and different volatilities. Allowing the market to have greater volatility, increases the probability that the stock will reach the strike price if the strike price is too high. Indeed, figure 9 shows that if the volatility is 20%, the option price is estimated adequately accurately if the strike price is less than $\$120$ which is 20% higher than the initial price. However, for higher strike prices, any ability to valuate the option is lost. If the volatility is raised to 50% we observe the same effect for strike prices greater than $\$150$.

Figure 10 examines the case of 20% volatility for two different numbers of trials. Using 10000 trials instead of 2500 indeed improves the fluctuations but the computational cost also increases.

The number of trials plays a minor role in the convergence of the option price versus the time steps as seen in figures 11 and 12. The time steps play the most important role in the convergence of the option price, while the number of trials affect the uncertainty of the result. The uncertainty cannot be completely eliminated for any number of trials because of the stochastic nature of the simulation.

Last but not least, we examine the importance of using the risk-free interest rate as the expected return (i.e., $\mu = r$). Figure 13 shows that the option price is correctly valuated only in the case where $\mu = r$. Otherwise, the price is either overestimated if $\mu > r$ or underestimated if $\mu < r$.

Figure 14 shows the relationship between the expected returns and the option price. As expected, higher expected returns increase the call option price while decreasing the price of the put option. The results from the BSM model and the simulation exhibit perfect agreement.

The price of the option versus the initial and the strike price are shown in figures 15 and 16 respectively. Again, the analytical and the simulated results show excellent agreement. The price of the call option increases (decreases) if the initial (strike) price increases because this creates a low risk scenario and, thus, a very desirable option. The opposite happens for the put option.

The loss is always limited to the amount invested but the profit can be huge if the stock price is much higher (lower) than the strike price for a call (put) option. For this to happen there must be great volatility and in this case the option prices should increase. Indeed, figure 17 shows that both options increase if the volatility is increased.

IV. Conclusion

The results of Monte Carlo simulations for pricing European options are in excellent agreement with the analytical solution from the Black-Scholes Merton equation. The effects of the different parameters of a Monte Carlo simulation were examined and the ones with the most profound impact were found to be the number of time steps and the number of trials.

The number of time steps is relevant in the convergence of the results while the number of trials affects the accuracy of the result. The uncertainty can never be completely eliminated due to the stochastic nature of the simulation. The computational cost increases both with the time steps and the number of trials. Thus, a trade-off between good convergence and accuracy and efficiency should be investigated before any calculation.

The strike price is a parameter strongly affected by the volatility on the market. The simulation performs well when the strike price is within the volatility limits. Also, the strike price affects the convergence versus the time steps. However, this effect is not that important since the time steps commonly used are more than enough to avoid it.

References

[1] P. P. Boyle, "Options: A Monte Carlo approach," *Journal of Financial Economics*, vol. 4, no. 3, pp. 323–338, 1977.

[2] F. Black and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, vol. 81, no. 3, pp. 637–654, 1973.

[3] R. C. Merton, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, vol. 4, pp. 141–183, 1973.

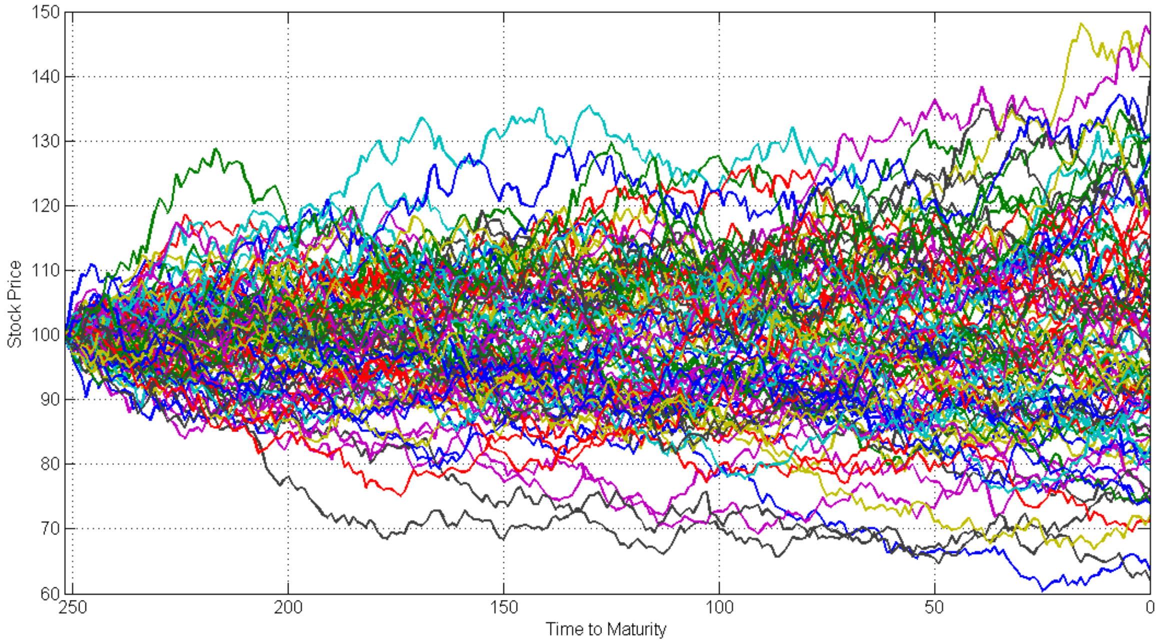


Figure 2: Typical output of the simulation for 1% drift

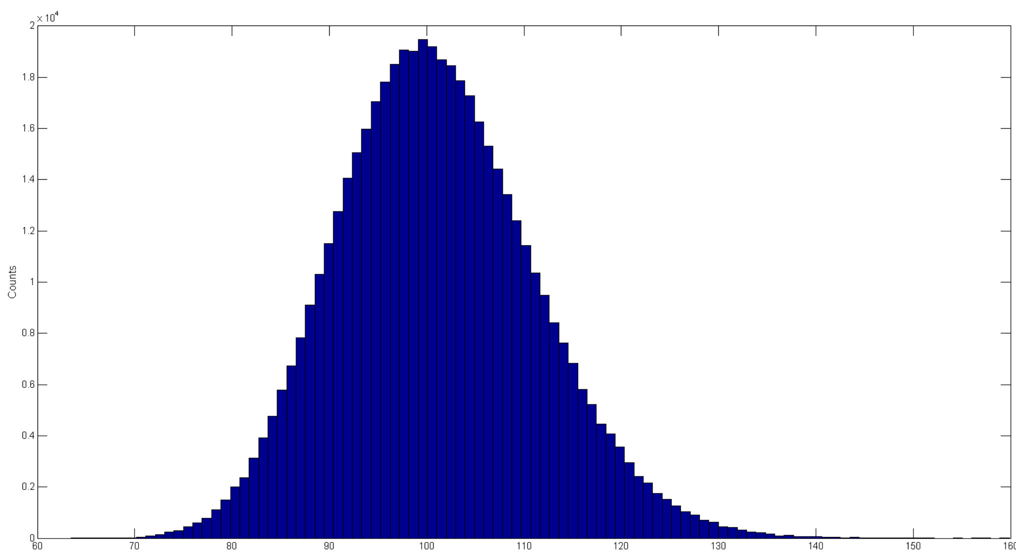


Figure 3: Lognormal distribution of the stock price at maturity

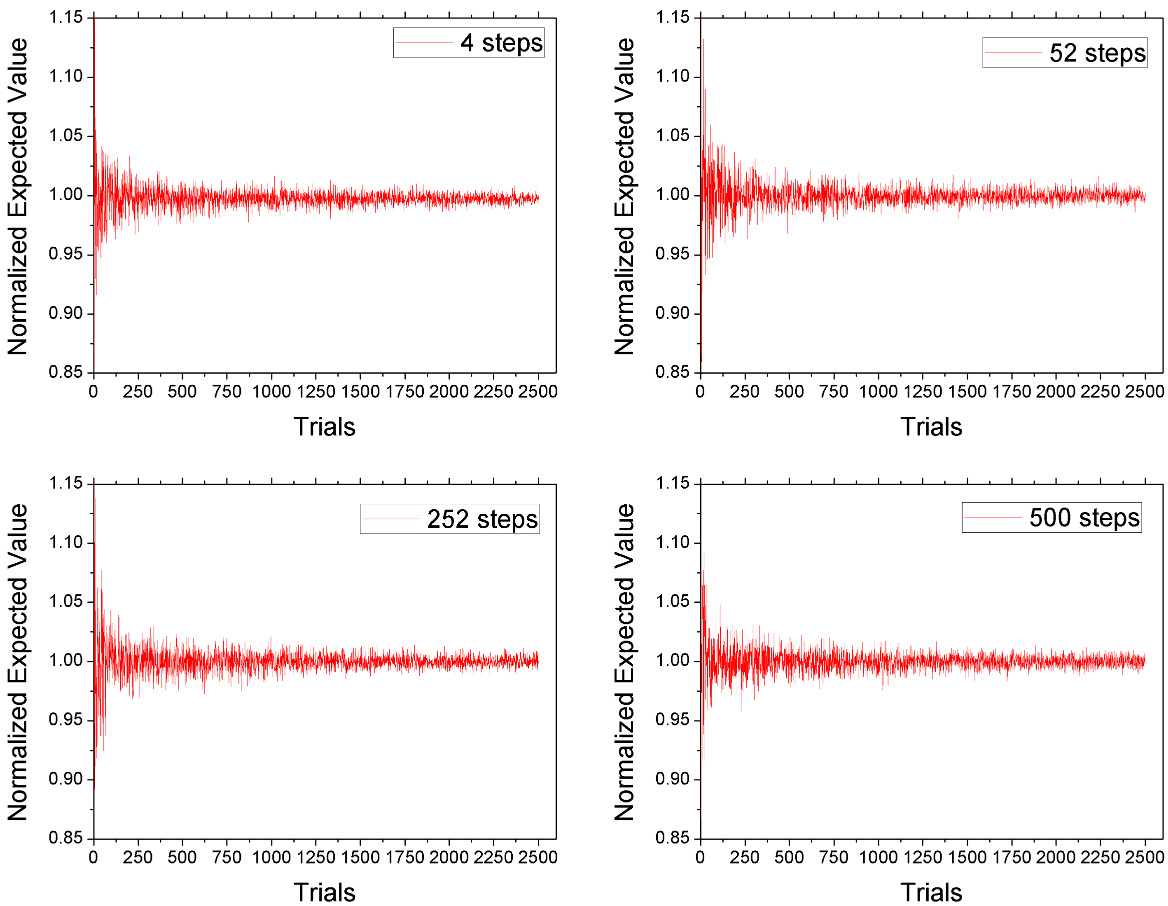


Figure 4: Time steps versus number of trials compared with $E(S_T) = S_0 e^{rT}$

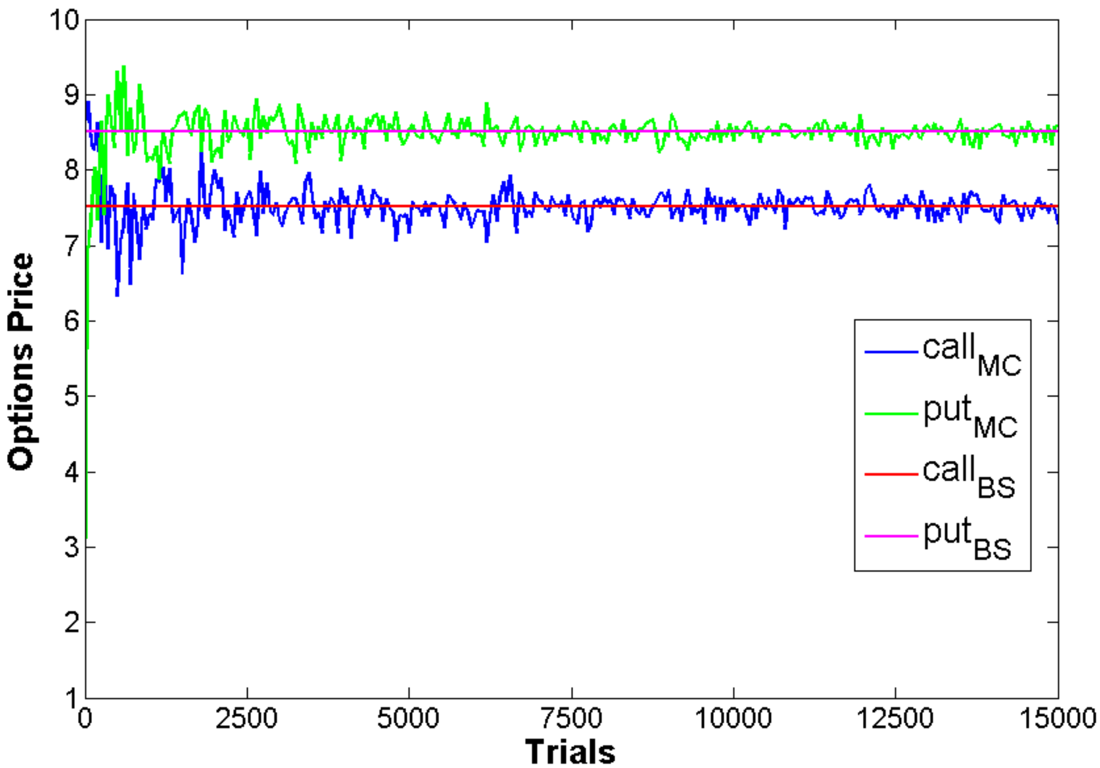


Figure 5: Price versus number of trials

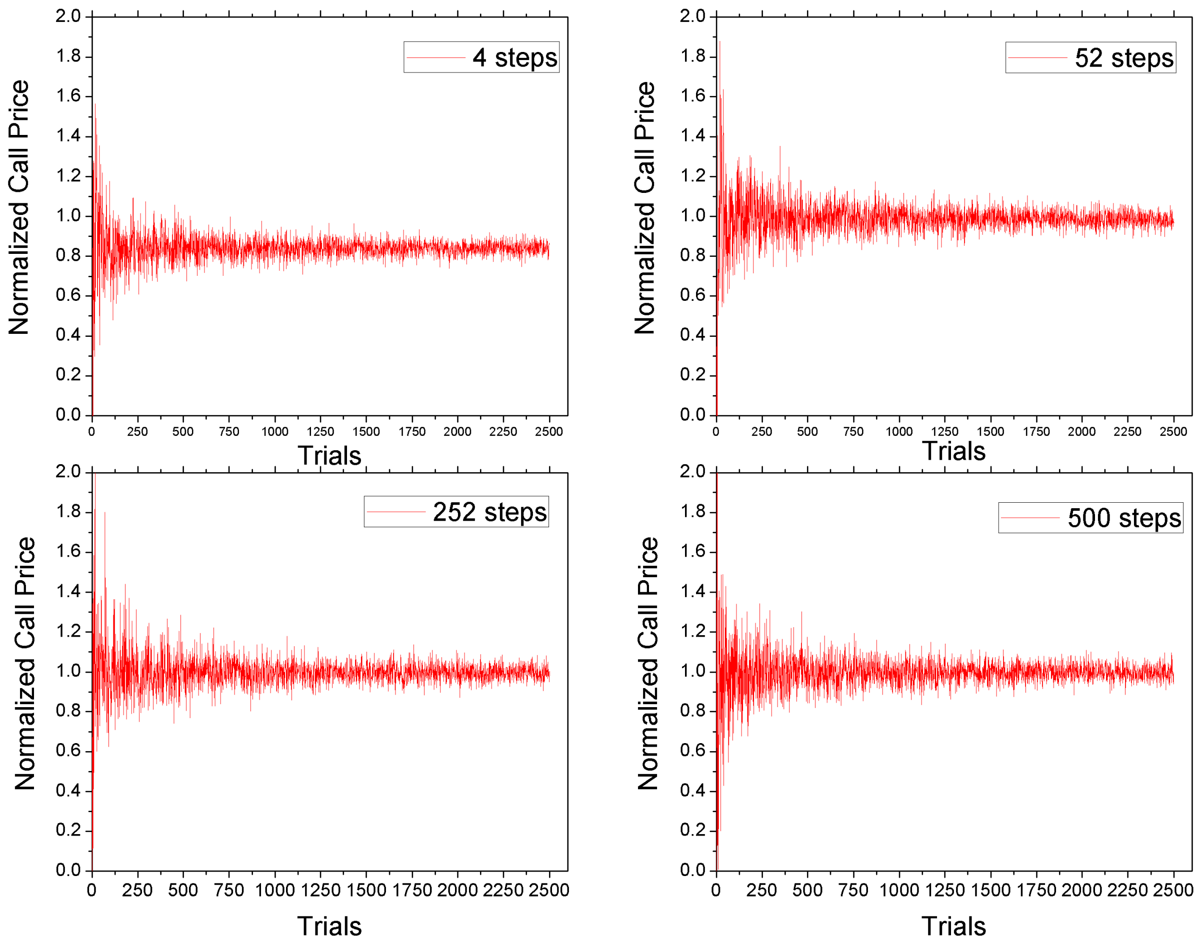


Figure 6: Time steps versus number of trials compared with the normalized call price

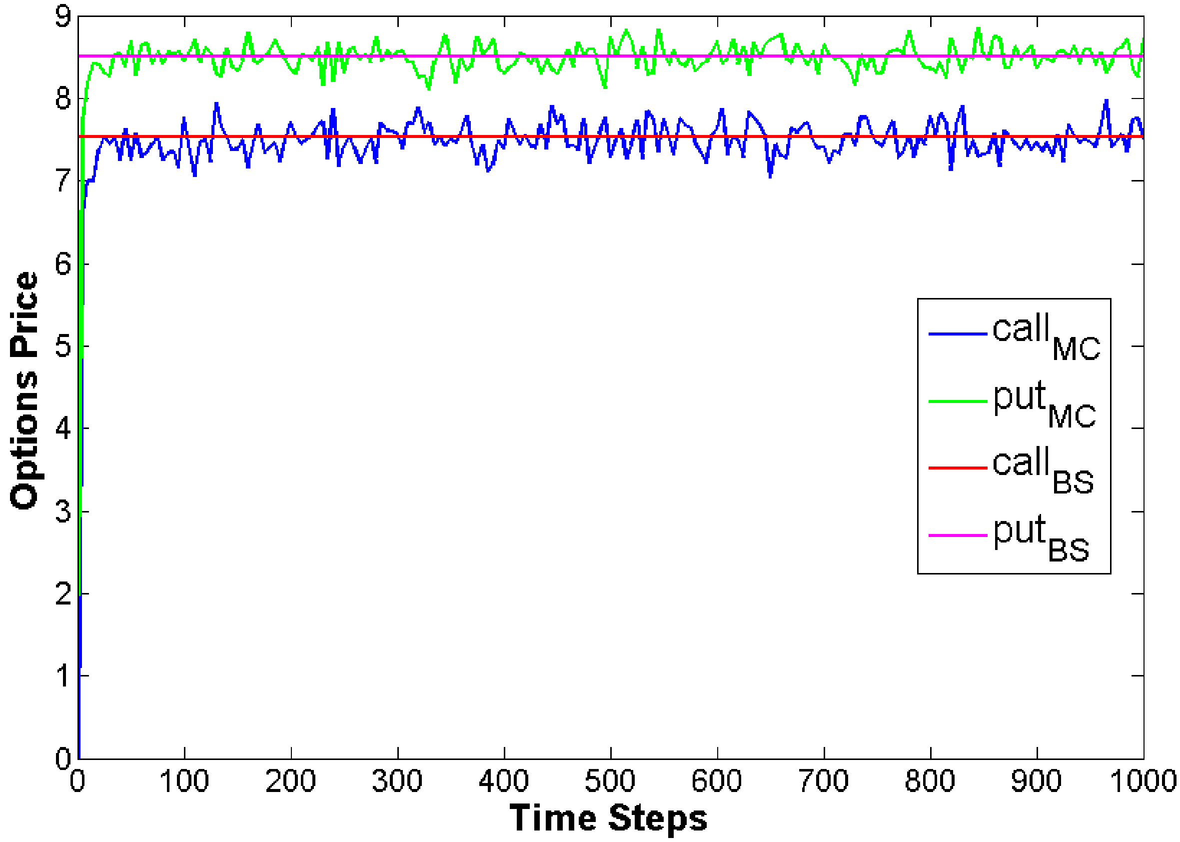


Figure 7: Options price versus time steps

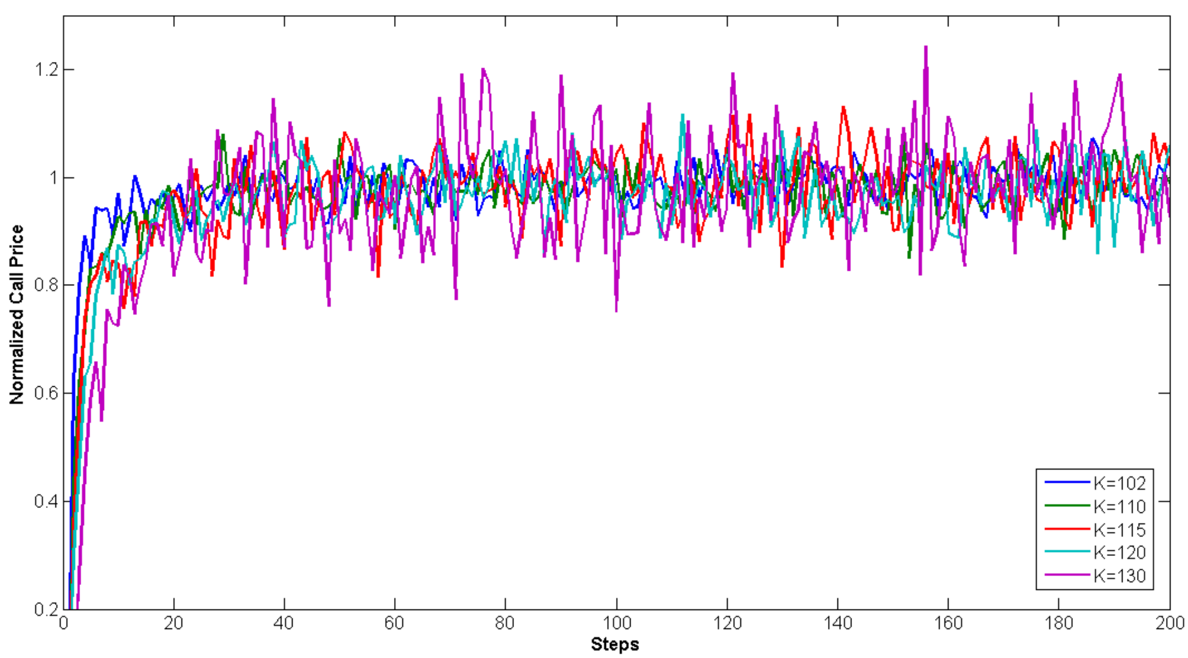


Figure 8: Normalized call price versus time steps for different K-values

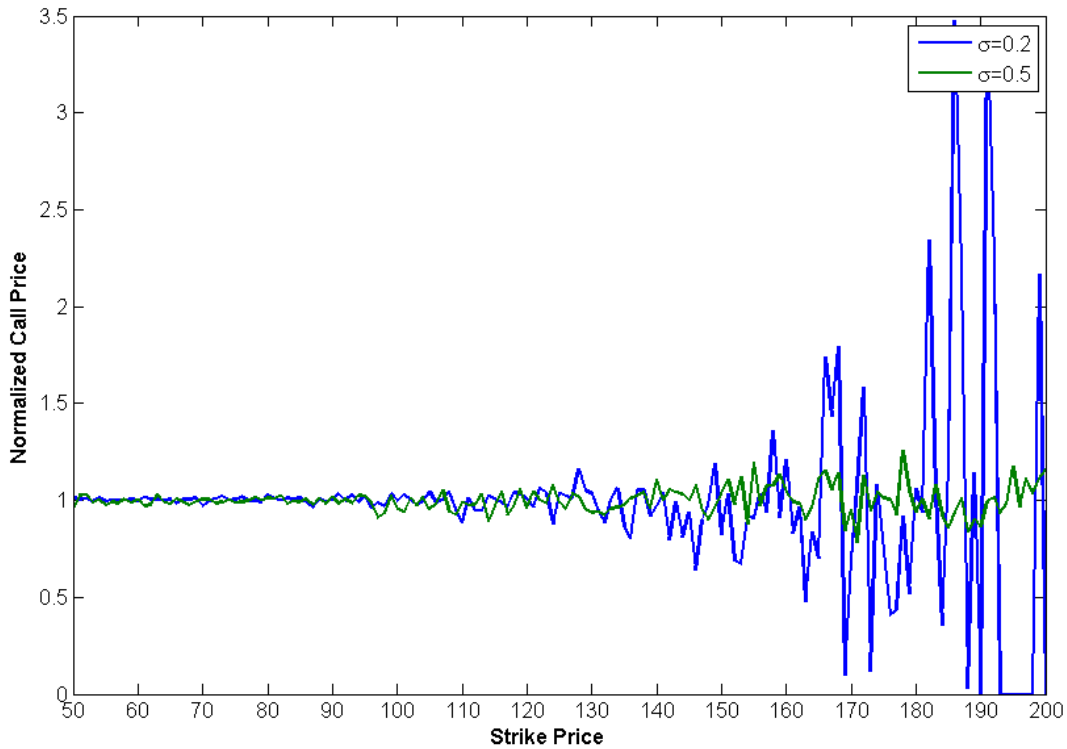


Figure 9: Normalized call price versus strike price

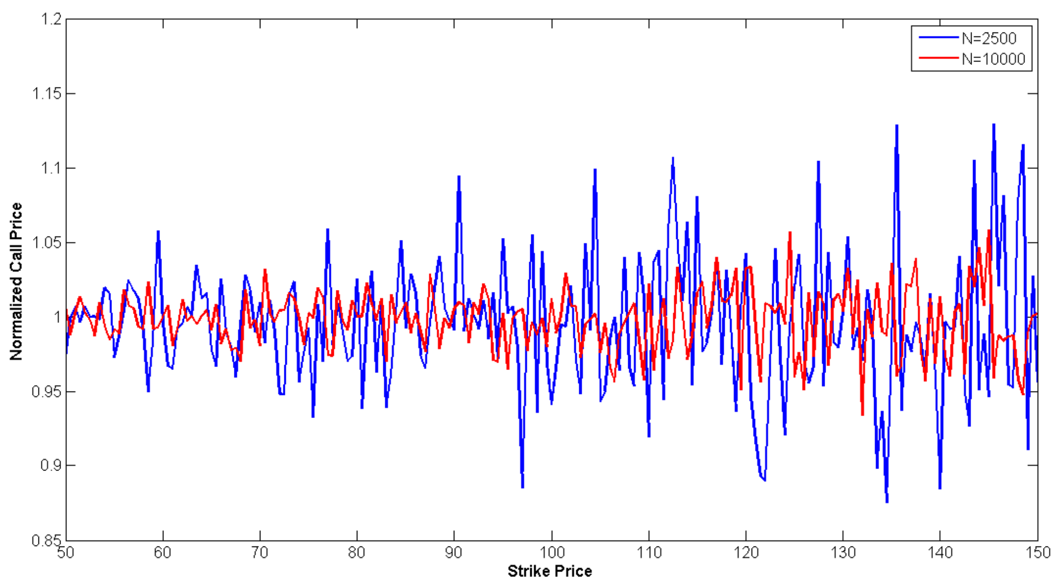


Figure 10: Normalized call price versus the strike price for different number of trials

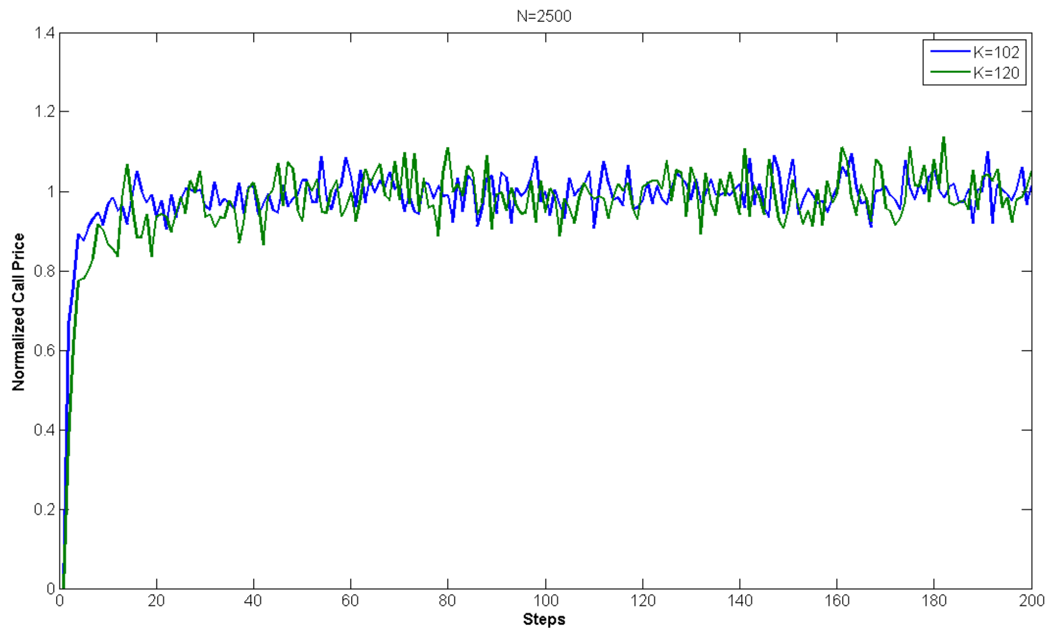


Figure 11: Normalized call price versus steps for different K values and 2500 trials

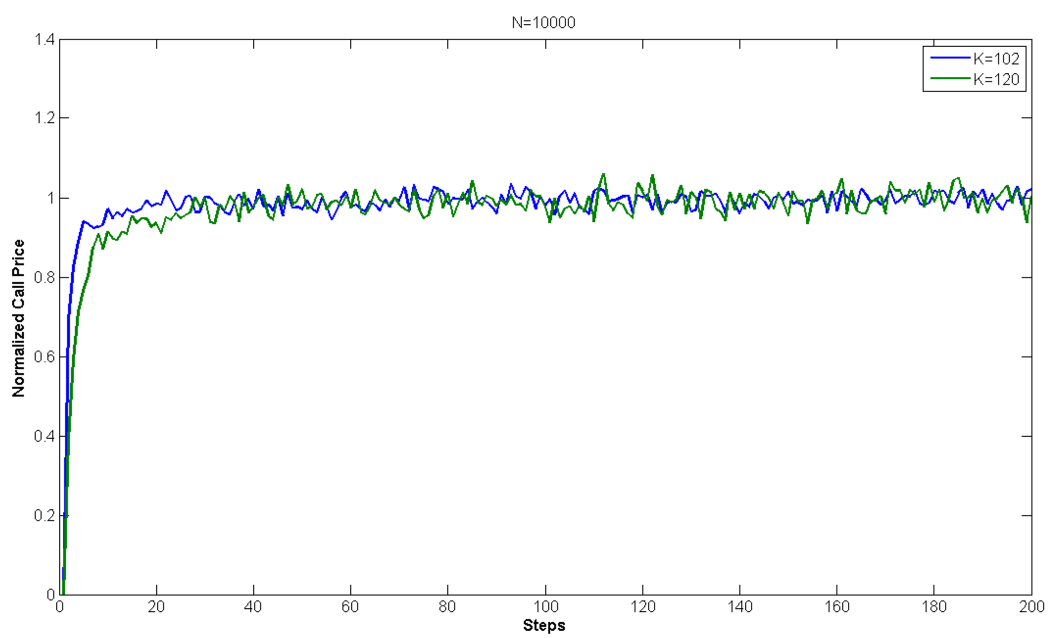


Figure 12: Normalized call price versus steps for different K values and 10000 trials

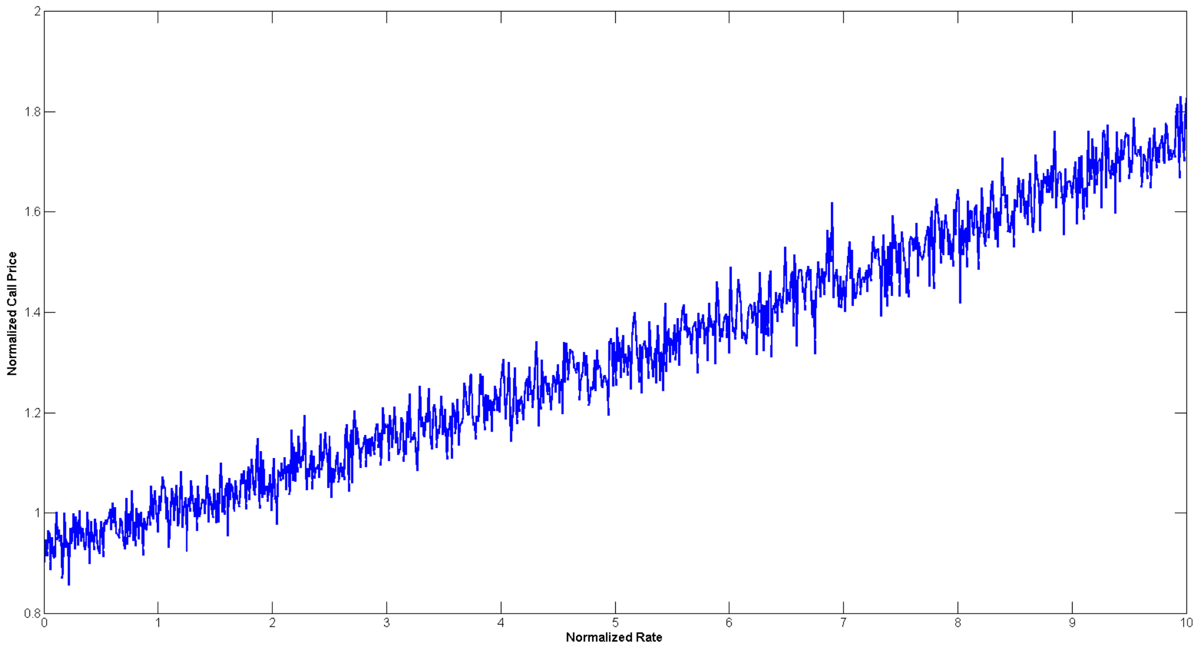


Figure 13: Normalized expected return rate versus risk-free rate (μ/r)

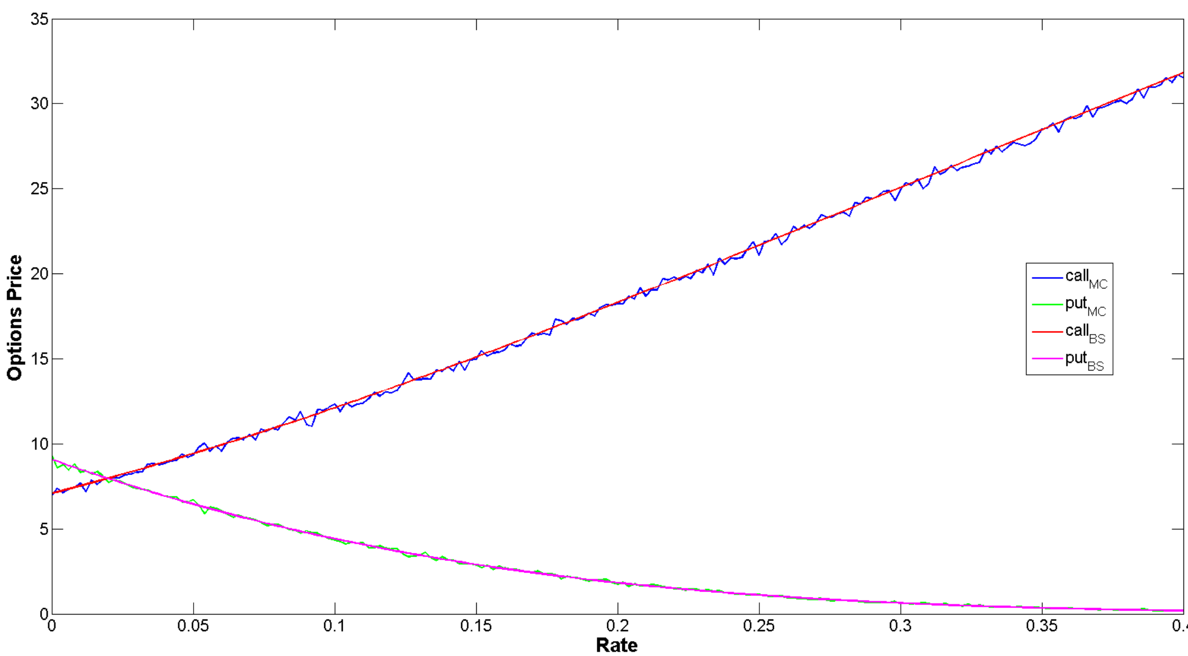


Figure 14: Options Price versus Return Rate

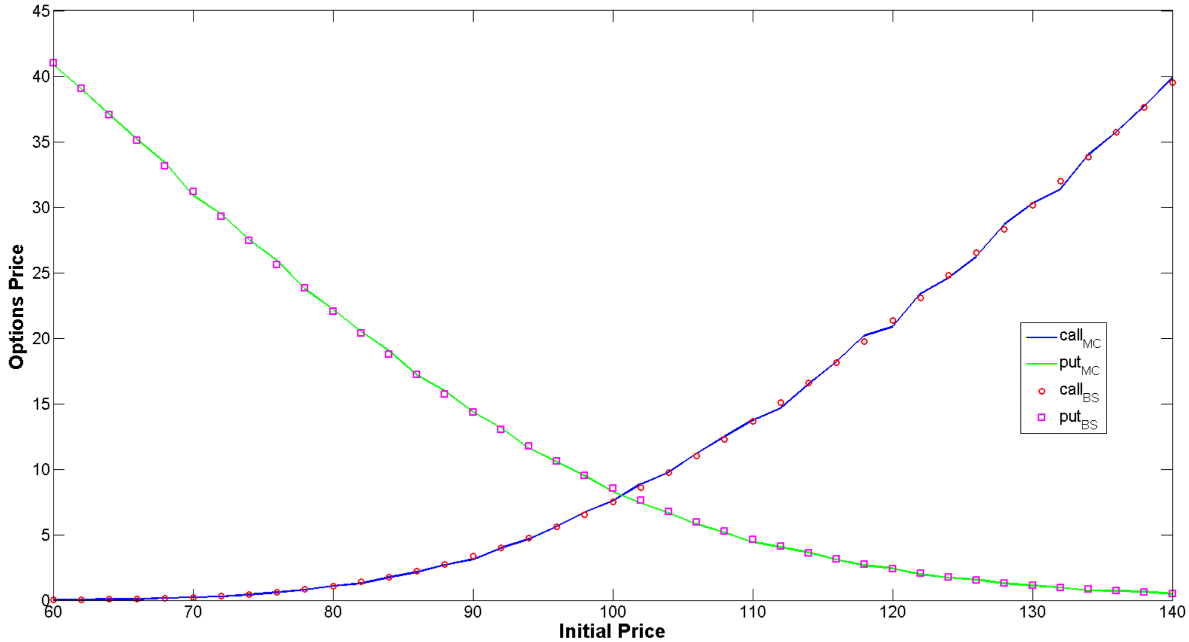


Figure 15: Options Price versus Initial Price

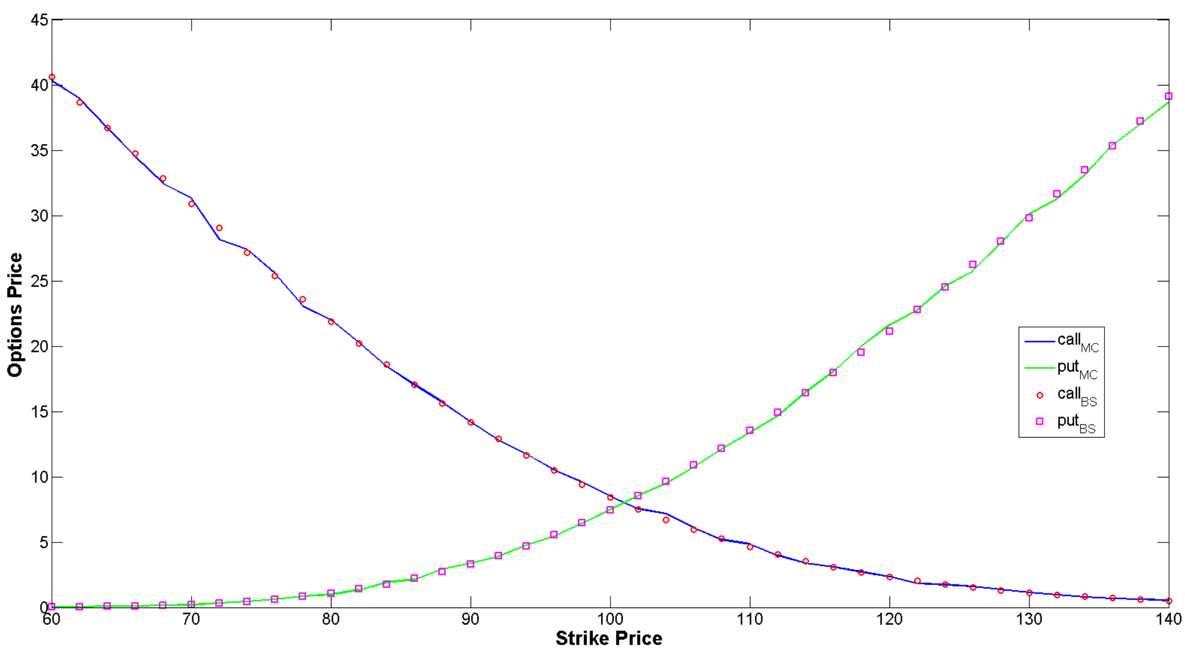


Figure 16: Options Price versus Strike Price

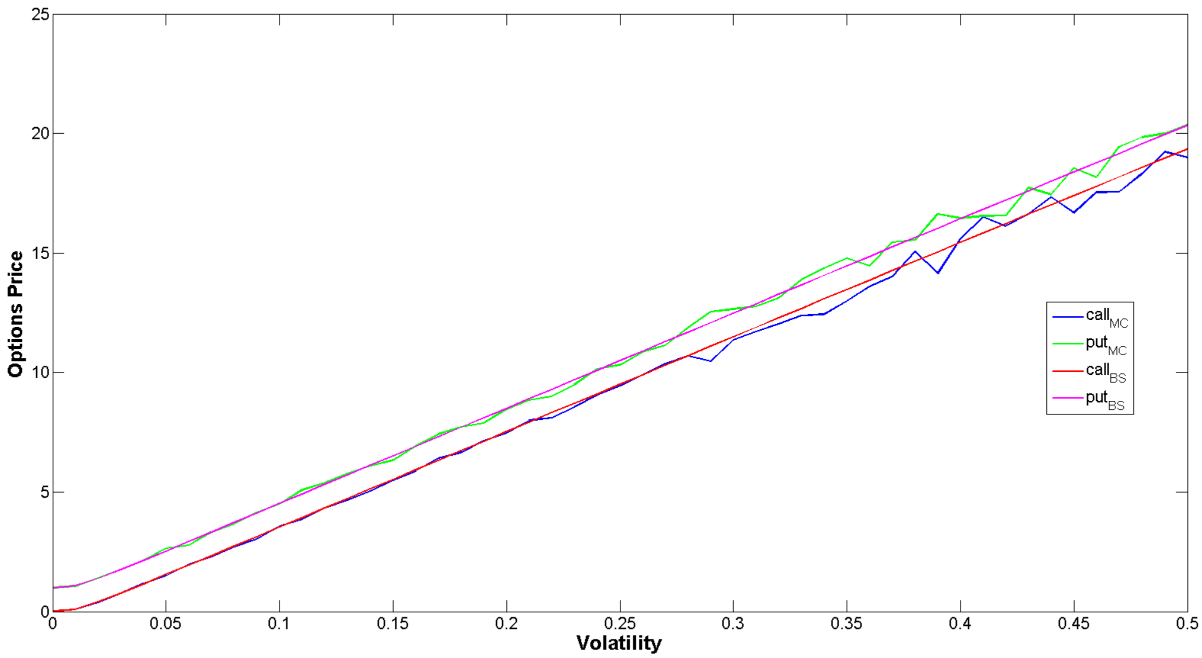


Figure 17: Options Price versus Volatility