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Efficient Analytic Approximation of American Option Values

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ABSTRACT

This paper provides simple, analytic approximations for pricing exchange-traded American call and put options written on commodities and commodity futures contracts. These approximations are accurate and considerably more computationally efficient than finite-difference, binomial, or compound-option pricing methods.

OPTIONS WRITTEN ON A wide variety of commodities and commodity futures contracts[1] now trade in the U.S. and Canada. Nearly all these options are American style[2] and thus have early exercise premiums implicitly embedded in their prices. Unlike the European-style option-pricing problems, however, analytic solutions for the American option-pricing problems have not been found, and the pricing of American options has usually resorted to finite-difference, binomial, or, more recently, compound-option approximation methods. While these approximation methods yield accurate American option values, they are cumbersome and expensive to use.

The purpose of this paper is to provide an accurate, inexpensive method for pricing American call and put options written on commodities and commodity futures contracts. The development of the "quadratic" approximation method is contained in Section I. Commodity option and commodity futures option contracts are defined, the underpinnings of commodity option valuation are discussed, and the solutions to the European call and put option-pricing problems are presented. Unlike the non-dividend-paying stock option case, it is shown that the American call option written on a commodity, as well as the American put option, may optimally be exercised prior to expiration. The approximation methods for the American call and put option values are then derived in the

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[1] Options on physical commodities (i.e., commodity options) were traded in the U.S. as early as the late 1800's. (See Mehl [14].) These options convey the right to buy or sell a certain physical commodity at a specified price within a specified period of time. The Commodity Exchange Act of 1936, however, banned trading in such options. Recently the CFTC introduced a pilot program allowing the various exchanges to reintroduce commodity options. Active trading now occurs not only in options on physical commodities such as gold and foreign currencies but also in options on commodity futures contracts (i.e., commodity futures options) such as wheat and livestock.

[2] The Chicago Board Options Exchange now lists European-style options on selected foreign currencies and the S&P 500 Composite Stock Index.

manner in which MacMillan [13] approximated the solution to the American put option on a non-dividend-paying stock pricing problem. In Section II, the programming of the approximations is considered, and the results of comparisons of the finite-difference, compound-option, and quadratic approximation methods are presented and discussed. Comparisons are also made to heuristic option-pricing methods. Section III contains a summary.

I. Valuation Equations for Commodity Options

In this section, the theory of pricing commodity and commodity futures option contracts is reviewed, and the approximations for the American call and put options are presented. At the outset it is useful to clearly define the terms "commodity option" and "commodity futures option." In the context in which the terms will be used here, a commodity option represents the right to buy or sell a specific commodity at a specified price within a specified period of time. The exact nature of the underlying commodity varies and may be anything from a precious metal such as gold or silver to a financial instrument such as a Treasury bond or a foreign currency. Usually the commodity option is labeled by the nature of the underlying commodity. For example, if the commodity option is written on a common stock, it is referred to as a "stock option," and, if the commodity option is written on a foreign currency, it is referred to as a "foreign currency option." If the underlying commodity is a futures contract, the options are referred to as "commodity futures options" or simply "futures options."

To begin, the focus will be on a general commodity option-pricing model. The assumptions used in the analysis are consistent with those introduced by Black and Scholes [3] and Merton [15]. First, the short-term interest rate, r, and the cost of carrying the commodity, b, are assumed to be constant, proportional rates. For a non-dividend-paying stock, the cost of carry is equal to the riskless rate of interest (i.e., $b = r$), but, for most other commodities, this is not the case. In Merton's [15] constant, proportional dividend-yield option-pricing models, for example, the cost of carrying the stock is the riskless rate, r, less the dividend yield, d (i.e., $b = r - d$). In Garman and Kohlhagen's [9] foreign currency option-pricing models, the cost of carrying the foreign currency is the domestic riskless rate, r, less the foreign riskless rate, r^* (i.e., $b = r - r^*$). However, that is not to say that the cost of carry is always below the riskless rate of interest. For the traditional agricultural commodities such as grain and livestock, the cost of carry exceeds the riskless rate by costs of storage, insurance, deterioration, etc.

In the absence of costless arbitrage opportunities, the assumption of a constant, proportional cost of carry suggests that the relationship between the futures and underlying commodity prices is

$$F = Se^{bT}, \tag{1}$$

where F and S are the current futures and spot prices, respectively, and T is the time to expiration of the futures contract. This relationship will prove useful later in this section.

A second common assumption in the option-pricing literature is that the underlying commodity price-change movements follow the stochastic differential equation,

$$dS/S = \alpha\, dt + \sigma\, dz, \tag{2}$$

where α is the expected instantaneous relative price change of the commodity, σ is the instantaneous standard deviation, and z is a Wiener process. It is worthwhile to note that, if the cost-of-carry relationship (1) holds and if equation (2) describes the movements of the commodity price through time, then the movements of the futures price are described by the equation,

$$dF/F = (\alpha - b)\, dt + \sigma\, dz. \tag{3}$$

That is, the expected instantaneous relative price change of the futures contract is $\alpha - b$ and the standard deviation of relative commodity price relatives is equal to the standard deviation of futures price relatives.[3]

Finally, assuming that a riskless hedge between the option and the underlying commodity may be formed, the partial differential equation governing the movements of the commodity option price (V) through time is

$$\tfrac{1}{2}\sigma^2 S^2 V_{SS} + bSV_S - rV + V_t = 0. \tag{4}$$

This equation, which first appeared in Merton [15], is the heart of the commodity option-pricing discussion contained herein. Note that, when the cost-of-carry rate b is equal to the riskless rate of interest, the differential equation (4) reduces to that of Black and Scholes [3], and, when the cost of carrying the underlying commodity is 0, the Black [2] commodity futures option differential equation is obtained. Both the non-dividend-paying stock and the futures option-pricing problems are special cases of this more general commodity option-pricing problem.

A. *European Commodity Options*

The differential equation (4) applies to calls and puts and to European options and American options. To derive the European call formula, the terminal boundary condition, $\max(0, S_T - X)$, is applied. Merton shows indirectly that, when this terminal boundary condition is applied to equation (4), the value of a European call option on a commodity is

$$c(S, T) = Se^{(b-r)T}\mathrm{N}(d_1) - Xe^{-rT}\mathrm{N}(d_2), \tag{5}$$

where $d_1 = [\ln(S/X) + (b + 0.5\sigma^2)T]/\sigma\sqrt{T}$, $d_2 = d_1 - \sigma\sqrt{T}$, and $\mathrm{N}(\cdot)$ is the cumulative univariate normal distribution.[4] When the lower boundary condition for the European put, $\max(0, X - S_T)$, is applied to the partial differential equation (4), the pricing equation is

$$p(S, T) = Xe^{-rT}\mathrm{N}(-d_2) - Se^{(b-r)T}\mathrm{N}(-d_1), \tag{6}$$

where all notation is as defined above.

[3] This result is noted in Black [2].

[4] It is interesting to note that substitution of the cost-of-carry relationship (1) into the European commodity formula (5) yields the Black [2] commodity futures option-pricing equation. This was first pointed out by Black and then later by Asay [1].

B. American Commodity Options

The European call formula (5) provides a convenient way of demonstrating that, under certain conditions, the American call option may be exercised early. Suppose $b < r$, as is the case with most of the non-common-stock commodity options traded. As the commodity price, S, becomes extremely large relative to the exercise price of the option, the values of $N(d_1)$ and $N(d_2)$ approach one and the European call value approaches $Se^{(b-r)T} - Xe^{-rT}$. However, the American option may be exercised immediately for $S - X$, which may be higher than the European option value when $b < r$. Thus, the American call option may command a higher price than the European call option because of the early exercise privilege. If $b \geq r$, as in the case of an option on a non-dividend-paying stock (i.e., $b = r$), the lower price bound of the European option will have a greater value than the exercisable proceeds of the American option for all levels of commodity price, so there is no possibility of early exercise and the European call option model (5) will accurately price American call options. For the American puts, there is always some possibility of early exercise, so the European formula (6) never applies. A more detailed explanation of the conditions for early exercise of the call and put options written on commodities is provided in Stoll and Whaley [19].

The valuation of American commodity options therefore involves addressing the early exercise feature of the options. When the American option boundary conditions are applied to (4), analytic solutions are not known and approximations must be used. The most common approach uses finite-difference methods. The first applications along these lines were by Schwartz [18], who valued warrants written on dividend-paying stocks, and by Brennan and Schwartz [5], who priced American put options on non-dividend-paying stocks.[5] Recently, Ramaswamy and Sundaresan [16] and Brenner, Courtadon, and Subrahmanyam [6] used finite-difference methods to price American options written on futures contracts.

The most serious limitation of using finite-difference methods to price American options is that they are computationally expensive. To ensure a high degree of accuracy, it is necessary to partition the commodity price and time dimensions into a very fine grid and enumerate every possible path the commodity option price could travel during its remaining time to expiration. This task is cumbersome and can only be efficiently accomplished with the use of a main-frame computer.

An alternative approximation method was recently introduced by Geske and Johnson [10]. Their compound-option approximation method is computationally less expensive than numerical methods and offers the advantages of being intuitively appealing and easily amenable to comparative-statics analysis. However, while being about twenty times more computationally efficient than numerical methods, the compound-option approach is still not inexpensive since it requires the evaluation of cumulative bivariate, trivariate, and sometimes higher

[5] Geske and Shastri [11] provide a comprehensive analysis of the merits of the explicit, implicit, and binomial finite-difference approximation methods. Cox, Ross, and Rubinstein [7] and Cox and Rubinstein [8] provide detailed discussions of the binomial option-pricing framework.

order multivariate normal density functions. Needless to say, such integral evaluations require the assistance of fairly sophisticated programs and are not practical on anything below the level of a fast microcomputer.

Johnson [12] and others provide heuristic techniques for valuing American put options on non-dividend-paying stocks. Although these techniques are very fast computationally, they are specific to the stock option-pricing problem and are not directly comparable to the general commodity option-pricing approximations discussed herein. In addition, the accuracy of heuristic techniques is frequently sensitive to the parameter range used in the option-pricing problem, a point that we will return to in the simulation results of the next section.

The American commodity option-pricing approximation method derived here is accurate, is amenable to comparative-statics analysis, and can be programmed on a hand-held calculator. The method is based on MacMillan's [13] quadratic approximation of the American put option on a non-dividend-paying stock valuation problem. To explain the derivation of our approximation, the problem of pricing an American call option on a commodity is addressed.

B.1. Quadratic Approximation of the American Call Value

The key insight into the quadratic approximation approach is that, if the partial differential equation (4) applies to American options as well as European options, it also applies to the early exercise premium of the American option. For an American call option written on a commodity, the early exercise premium $\varepsilon_C(S, T)$ is defined as

$$\varepsilon_C(S, T) = C(S, T) - c(S, T), \tag{7}$$

where $C(S, T)$ is the American commodity option value and $c(S, T)$ is the European commodity option value as described by equation (5). The partial differential equation for the early exercise premium is therefore

$$\tfrac{1}{2}\sigma^2 S^2 \varepsilon_{SS} - r\varepsilon + bS\varepsilon_S + \varepsilon_t = 0. \tag{8}$$

For ease of exposition, two simplifications are made. First, in place of time t evolving from the present toward the option's expiration t^*, time T evolving from the option's expiration to the present, that is, $T = t^* - t$, is used. Thus, $\varepsilon_T = -\varepsilon_t$. Second, equation (8) is multiplied by $2/\sigma^2$, and, third, the notational substitutions $M = 2r/\sigma^2$ and $N = 2b/\sigma^2$ are made. Equation (8) now reads as

$$S^2\varepsilon_{SS} - M\varepsilon + NS\varepsilon_S - (M/r)\varepsilon_T = 0. \tag{9}$$

The early exercise premium is then defined as $\varepsilon_C(S, K) = K(T)f(S, K)$. It therefore follows that $\varepsilon_{SS} = Kf_{SS}$ and $\varepsilon_T = K_T f + KK_T f_K$. Substituting the partial derivative expressions into (9), factoring K, and gathering terms on Mf yield

$$S^2 f_{SS} + NSf_S - Mf[1 + (K_T/rK)(1 + Kf_K/f)] = 0. \tag{10}$$

Choosing $K(T) = 1 - e^{-rT}$, substituting into (10), and simplifying give

$$S^2 f_{SS} + NSf_S - (M/K)f - (1 - K)Mf_K = 0. \tag{11}$$

Up to this point, the analysis has been exact, and no approximation has been made. The approximation will be made in equation (11); the last term on the left-hand side will be assumed to be equal to 0. For commodity options with very short (long) times to expiration, this assumption is reasonable since, as T approaches 0 (∞), f_K approaches 0 (K approaches 1), and the term, $(1 - K)Mf_K$, disappears. As an approximation, therefore, the last term is dropped, and the approximation of the early exercise premium differential equation is

$$S^2 f_{SS} + NSf_S - (M/K)f = 0. \tag{12}$$

Equation (12) is a second-order ordinary differential equation with two linearly independent solutions of the form aS^q. They can be found by substituting $f = aS^q$ into (12):

$$aS^q[q^2 + (N - 1)q - M/K] = 0. \tag{13}$$

The roots of (13) are $q_1 = [-(N - 1) - \sqrt{(N - 1)^2 + 4M/K}]/2$ and $q_2 = [-(N - 1) + \sqrt{(N - 1)^2 + 4M/K}]/2$. Note that, because $M/K > 0$, $q_1 < 0$ and $q_2 > 0$.

The general solution to (12) is

$$f(S) = a_1 S^{q_1} + a_2 S^{q_2}. \tag{14}$$

With q_1 and q_2 known, a_1 and a_2 are left to be determined. With $q_1 < 0$ and $a_1 \neq 0$, the function f approaches ∞ as the commodity price S approaches 0. This is unacceptable since the early exercise premium of the American call becomes worthless when the commodity price drops to zero. The first constraint to be imposed is, therefore, $a_1 = 0$, and the approximate value of the American call option written on a commodity will be written as

$$C(S, T) = c(S, T) + Ka_2 S^{q_2}. \tag{15}$$

To find an appropriate constraint on a_2, consider equation (15). As $S = 0$, $C(S, T) = 0$ since both $c(S, T)$ and $Ka_2S^{q_2}$ are equal to 0. As S rises, the value of $C(S, T)$ rises for two reasons: $c(S, T)$ rises and $Ka_2S^{q_2}$ rises, assuming $a_2 > 0$. In order to represent the value of the American call, however, the function on the right-hand side of (15) should touch, but not intersect, the boundary imposed by the early exercise proceeds of the American call, $S - X$. Below the critical commodity price S^* implied by the point of tangency, the American call value is represented by equation (15). Above S^*, the American call value is equal to its exercisable proceeds, $S - X$, and the fact that $a_2S^{q_2}$ rises at a faster and faster rate above S^* is not of concern.

To find the critical commodity price S^*, the exercisable value of the American call is set equal to the value of $C(S^*, T)$ as represented by (15), that is,

$$S^* - X = c(S^*, T) + Ka_2 S^{*q_2}, \tag{16}$$

and the slope of the exercisable value of the call, one, is set equal to the slope of $C(S^*, T)$, that is,

$$1 = e^{(b-r)T}\mathrm{N}[d_1(S^*)] + Kq_2a_2S^{*q_2-1}, \tag{17}$$

where $e^{(b-r)T}\mathrm{N}[d_1(S^*)]$ is the partial derivative of $c(S^*, T)$ with respect to S^* and where $d_1(S^*) = [\ln(S^*/X) + (b + 0.5\sigma^2)T]/\sigma\sqrt{T}$. Thus, there are two equations, (16) and (17), and two unknowns, a_2 and S^*. Isolating a_2 in (17) yields

$$a_2 = \{1 - e^{(b-r)T}\mathrm{N}[d_1(S^*)]\}/Kq_2S^{*q_2-1}. \tag{18}$$

Substituting (18) into (16) and simplifying results in a critical commodity price, S^*, that satisfies

$$S^* - X = c(S^*, T) + \{1 - e^{(b-r)T}\mathrm{N}[d_1(S^*)]\}S^*/q_2. \tag{19}$$

Although S^* is the only unknown value in equation (19), it must be determined iteratively. An efficient algorithm for finding S^* is presented in the next section. With S^* known, equation (16) provides the value of a_2. Substituting (18) into (15) and simplifying yields

$$\begin{aligned} C(S, T) &= c(S, T) + A_2(S/S^*)^{q_2}, && \text{when } S < S^*, \text{ and} \\ C(S, T) &= S - X, && \text{when } S \geq S^*, \end{aligned} \tag{20}$$

where $A_2 = (S^*/q_2)\{1 - e^{(b-r)T}\mathrm{N}[d_1(S^*)]\}$. Note that $A_2 > 0$ since S^*, q_2, and $1 - e^{(b-r)T}\mathrm{N}[d_1(S^*)]$ are positive when $b < r$. Equation (20) is therefore an efficient analytic approximation of the value of an American call option written on a commodity when the cost of carry is less than the riskless rate of interest. When $b \geq r$, the American call will never be exercised early, and valuation equation (5) applies.[6]

In equation (20), it is worthwhile to note that the early exercise premium of the American call option on a commodity approaches 0 as the time to expiration of the option approaches 0. As T gets small, $\mathrm{N}[d_1(S^*)]$ approaches 1,[7] $\{1 - e^{(b-r)T}\mathrm{N}[d_1(S^*)]\}$ approaches 0, A_2 approaches 0, and, thus, $A_2(S/S^*)^{q_2}$ approaches 0.

B.2. Quadratic Approximation of the American Put Value

Before proceeding with a discussion of how to use this quadratic approximation, it is useful to note how the approximation would change for the American put option on a commodity. Since the partial differential equation (8) applies to the early exercise premium of the American put

$$\varepsilon_p(S, T) = P(S, T) - p(S, T), \tag{21}$$

equations (9) through (14) of the analysis remain the same. In (14), it is now the term $a_1S^{q_1}$ that is of interest since the early exercise premium of the American put must approach 0 as S approaches positive infinity. The term, $a_2S^{q_2}$, violates this boundary condition, so a_2 is set equal to zero and the approximate value of the American put option becomes

$$P(S, T) = p(S, T) + Ka_1S^{q_1}. \tag{22}$$

Again, the values of the coefficient a_1 and the critical commodity price S^{**} must be determined, and the necessary steps pattern those used in determining a_2 and S^*. The value of a_1 is

[6] In coding the American call option-pricing algorithm, first check whether the cost of carry b is less than the riskless rate of interest r. If not, then price the call using the European pricing formula.

[7] The term $\mathrm{N}[d_1(S^*)]$ approaches 1 rather than 0 as T approaches 0 because the critical commodity price S^* is always greater than or equal to the exercise price X.

$$a_1 = -\{1 - e^{(b-r)T}\mathrm{N}[-d_1(S^{**})]\}/Kq_1S^{**q_1-1}, \tag{23}$$

where $-e^{(b-r)T}\mathrm{N}[-d_1(S^{**})]$ is the partial derivative of $p(S^{**}, T)$ with respect to S^{**} and where $a_1 > 0$ since $q_1 < 0$ and since all other terms are positive. The critical commodity price S^{**} is determined by solving

$$X - S^{**} = p(S^{**}, T) - \{1 - e^{(b-r)T}N[-d_1(S^{**})]\}S^{**}/q_1. \tag{24}$$

With S^{**} known, the approximate value of an American put option written on a commodity (22) becomes

$$P(S, T) = p(S, T) + A_1(S/S^{**})^{q_1}, \quad \text{when } S > S^{**}, \text{ and}$$
$$P(S, T) = X - S, \qquad \text{when } S \leq S^{**}, \tag{25}$$

where $A_1 = -(S^{**}/q_1)\{1 - e^{(b-r)T}\mathrm{N}[-d_1(S^{**})]\}$. Note that $A_1 > 0$ since $q_1 < 0$, $S^{**} > 0$, and $\mathrm{N}[-d_1(S^{**})] < e^{-bT}$.

C. American Commodity Futures Options

Up to this point, the focus of the discussion has been on the valuation of commodity options where the cost of carrying the underlying commodity is a constant, proportional rate b. If b is set equal to certain specific values, however, specific commodity option-valuation equations are obtained. For example, the cost of carrying any futures position is equal to 0. Thus, to obtain the commodity futures option-valuation results, simply set b equal to zero in the approximation just described. The approximate value of an American call option on a futures contract is given by equation (20), where the futures price, F, is substituted for the commodity price, S, and where the cost of carry, b, is set equal to zero. The approximate value of an American put option on a futures contract is given by equation (25), where similar substitutions are made. Both of these American futures option-price approximations are used in Whaley [21].

D. American Stock Options

Another special case of the commodity option-valuation framework is the non-dividend-paying stock option. The cost of carrying the underlying stock is assumed to be equal to the riskless rate of interest; in other words, b is set equal to r in the above approximation. It is worthwhile to point out that, since $b = r$ for this option-pricing problem, the American call will be valued using the European formula (5). The resulting approximation for the American put is that of MacMillan [13].

E. Summary

The quadratic approximation techniques for pricing the American call and put options on a commodity have now been derived. Before presenting some simulation results intended to show the accuracy of the techniques, it is worthwhile to reiterate that they are useful in a wide range of option-pricing problems. The futures option and the stock option cases are only two examples. American options on foreign currencies, on stock indexes with continuous dividend yields, on precious metals such as gold and silver, and on long-term debt instruments with continuous coupon yields can be accurately priced within this framework.

II. Implementation and Simulation of Approximation Method

In the approximation procedure outlined in the last section, only one step, the determination of the critical commodity price S^*, is not straightforward. In this section, an efficient algorithm for determining S^* is presented, and then simulated results from the quadratic approximation method are compared with results for the finite-difference and compound-option approximation methods.

A. An Algorithm for Determining S^*

To find the critical commodity price S^*, it is necessary to solve equation (19). Since this cannot be done directly, an iterative procedure must be developed.[8] To begin, evaluate both sides of equation (19) at some seed value, S_1, that is,

$$\text{LHS}(S_i) = S_i - X, \quad \text{and} \tag{26a}$$

$$\text{RHS}(S_i) = c(S_i, T) + \{1 - e^{(b-r)T}\text{N}[d_1(S_i)]\}S_i/q_2, \tag{26b}$$

where $d_1(S_i) = [\ln(S_i/X) + (b + 0.5\sigma^2)T]/\sigma\sqrt{T}$ and $i = 1$. Naturally, it is unlikely that $\text{LHS}(S_i) = \text{RHS}(S_i)$ on the initial guess of S_1, and a second guess must be made. To develop the next guess S_{i+1}, first find the slope b_i of the RHS at S_i, that is,

$$b_i = e^{(b-r)T}\text{N}[d_1(S_i)](1 - 1/q_2) + [1 - e^{(b-r)T}\text{n}[d_1(S_i)]/\sigma\sqrt{T}]/q_2, \tag{27}$$

where $\text{n}(\cdot)$ is the univariate normal density function. Next, find where the line tangent to the curve RHS at S_i intersects the exercisable proceeds of the American call, $S - X$, that is,

$$\text{RHS}(S_i) + b_i(S - S_i) = S - X,$$

and then isolate S to find S_{i+1},

$$S_{i+1} = [X + \text{RHS}(S_i) - b_iS_i]/(1 - b_i). \tag{28}$$

Equation (28) will provide the second and subsequent guesses of S, with new values of (26a), (26b), (27), and (28) computed with each new iteration. The iterative procedure should continue until the relative absolute error falls within an acceptable tolerance level; for example,

$$|\text{LHS}(S_i) - \text{RHS}(S_i)|/X < 0.00001. \tag{29}$$

B. Seed Value

The iterative technique outlined here converges reasonably quickly by setting the seed value S_1 equal to the option's exercise price X and by imposing the

[8] Iterative solution to (19) for each of the options on a single underlying commodity is unnecessary. The critical commodity price in (19) is proportional in X; thus, if the critical commodity price S_1^* is computed for an option with exercise price X_1, the critical commodity price for a second option with a different exercise price X_2 is simply $S_2^* = (S_1^*/X_1)X_2$.

tolerance criterion (29). The speed with which the algorithm finds the critical commodity price, however, can be improved by using a starting point closer to the solution.

To arrive at an approximate value of the critical commodity price, consider the information contained in equation (19). If the time to expiration of the call option is equal to 0, the critical commodity price above which the option will be exercised is the exercise price of the option, X. At the other extreme, if the time remaining to expiration is infinite, the critical commodity price may be solved exactly by substituting $T = +\infty$ in (19), that is,

$$S^*(\infty) = X/[1 - 1/q_2(\infty)], \tag{30}$$

where $q_2(\infty) = [-(N - 1) + \sqrt{(N - 1)^2 + 4M}]/2$. Equation (19) also shows that the critical commodity price is an increasing function of time to expiration of the option.

With this and other information from the call option-pricing problem in hand, it is possible to derive an approximate analytic solution to finding the critical commodity price. Such a derivation is provided in the Appendix. The final form of the approximation is

$$S^* = X + [S^*(\infty) - X][1 - e^{h_2}], \tag{31}$$

where $h_2 = -(bT + 2\sigma\sqrt{T})\{X/[S^*(\infty) - X]\}$. Note that (31) satisfies the critical commodity-price restrictions when $T = 0$ and $T = +\infty$.

For the put option-pricing problem, the critical commodity price must satisfy equation (24). At $T = 0$, the critical price is again the exercise price of the option, and, at $T = +\infty$,

$$S^{**}(\infty) = X/[1 - 1/q_1(\infty)], \tag{32}$$

where $q_1(\infty) = [-(N - 1) - \sqrt{(N - 1)^2 + 4M}]/2$. It is worthwhile to point out that, when the cost of carry b is equal to the riskless rate of interest r, this result is exactly the same as Merton's [15]. In equation (24), the critical commodity price is a decreasing function of time to expiration, and an approximate analytic expression for the critical commodity price is

$$S^{**} = S^{**}(\infty) + [X - S^{**}(\infty)]e^{h_1}, \tag{33}$$

where $h_1 = (bT - 2\sigma\sqrt{T})\{X/[X - S^{**}(\infty)]\}$.[9]

Equations (31) and (33) provide the seed values for the iterative procedures that determine the critical commodity price in the American call and the American put option algorithms. Both are straightforward computations, and their use usually ensures convergence in three iterations or less.

C. Simulation Results

Tables I through IV contain a sensitivity analysis of the theoretical European and American commodity option values for a variety of cost-of-carry parameters.

[9] For very large values of b and T, the influence of b must be bounded in the put exponent to ensure critical prices monotonically decreasing in T. A reasonable upper bound on b is $0.6\sigma/\sqrt{T}$, so the critical commodity price declines at least with a velocity $e^{-1.4\sigma}\sqrt{T}$.

Table I

Theoretical American Commodity Option Values Using Finite-Difference, Compound-Option, and Quadratic Approximation Methods (Cost of Carry (b) = −0.04 and Exercise Price (X) = 100)

		Call Options				Put Options			
			American $C(S,T)$				American $P(S,T)$		
Option Parameters[a]	Commodity Price S	European $c(S,T)$[b]	Finite-Difference Method[c]	Compound-Option Method[d]	Quadratic Approximation Method[e]	European $p(S,T)$[b]	Finite-Difference Method[c]	Compound-Option Method[d]	Quadratic Approximation Method[e]
	80	0.03	0.03	0.03	0.03	20.41	20.41	20.41	20.42
$r = 0.08$, $\sigma = 0.20$, $T = 0.25$	90	0.57	0.58	0.58	0.59	11.25	11.25	11.25	11.25
	100	3.42	3.52	3.52	3.52	4.40	4.40	4.40	4.40
	110	9.85	10.35	10.38	10.31	1.12	1.12	1.12	1.12
	120	18.62	20.00	19.97	20.00	0.18	0.19	0.18	0.18
	80	0.03	0.03	0.03	0.03	20.21	20.23	20.23	20.25
$r = 0.12$, $\sigma = 0.20$, $T = 0.25$	90	0.56	0.58	0.57	0.59	11.14	11.14	11.14	11.15
	100	3.39	3.50	3.49	3.51	4.35	4.35	4.35	4.35
	110	9.75	10.32	10.36	10.29	1.11	1.11	1.11	1.11
	120	18.43	20.00	19.97	20.00	0.18	0.18	0.18	0.18
	80	1.05	1.06	1.05	1.07	21.43	21.44	2.144	21.46
$r = 0.08$, $\sigma = 0.40$, $T = 0.25$	90	3.23	3.27	3.27	3.28	13.91	13.91	13.92	13.93
	100	7.29	7.40	7.41	7.41	8.27	8.26	8.27	8.27
	110	13.25	13.52	13.51	13.50	4.52	4.52	4.52	4.52
	120	20.73	21.29	21.30	21.23	2.29	2.29	2.29	2.30
	80	0.21	0.21	0.21	0.23	20.95	20.96	20.96	20.98
$r = 0.08$, $\sigma = 0.20$, $T = 0.50$	90	1.31	1.36	1.36	1.39	12.63	12.63	12.63	12.64
	100	4.46	4.71	4.69	4.72	6.37	6.37	6.37	6.37
	110	10.16	11.00	11.03	10.96	2.65	2.65	2.65	2.65
	120	17.85	20.00	19.98	20.00	0.92	0.92	0.92	0.92

[a] The notation in this column is as follows: r = riskless rate of interest; σ = standard deviation of the commodity price-change relative; and T = time to expiration.

[b] Values are computed using equations (5) and (6).

[c] Values are computed using the implicit finite-difference method with commodity price steps of 0.10 and time steps of 0.20 days or 0.0005479 years.

[d] Values are computed using the three-point extrapolation of the compound-option valuation approach.

[e] Values are computed using the quadratic approximation equations (20) and (25).

In Tables I and II, for example, the cost-of-carry parameter (b) is set equal to -0.04 and 0.04, respectively. Thus, the values in these tables may be thought of as being American foreign currency option prices, where the foreign riskless rate of interest is greater than and less than the domestic interest rate, respectively. In Table III, the cost-of-carry parameter is set equal to 0, so the resulting option values are for American commodity futures options. Finally, in Table IV, the cost of carry is set equal to the riskless rate of interest. Since this is the non-dividend-paying stock option case, only American put option values are reported.[10]

In the first three tables, three methods for pricing the American commodity options are used: (a) the implicit finite-difference approximation method with commodity price steps of 0.10 and time steps of 0.20 days or 0.0005479 years, (b) the compound-option approximation method using a three-point extrapolation, and (c) the quadratic approximation method. The European model values are included to provide an indication of the magnitude of the early exercise premium on American options. In the fourth table, the values of Johnson's [12] heuristic technique are also provided.

C.1. *Commodity Option Results*

Judging by the results reported in Tables I and II, the quadratic approximation is very accurate. The option prices for this method are within pennies of the implicit finite-difference method.[11] The most extreme errors occur for the in-the-money options where the volatility parameter is set equal to 0.40 and where the cost of carry is -0.04 for the calls and 0.04 for the puts (see Tables I and II), but even there the degree of mispricing, when compared with the finite-difference method, is less than three tenths of one percent. Considering that the quadratic approximation costs roughly 2000 times less, this result is impressive.

The compound-option valuation method appears to do about as well as the quadratic approximation at pricing American options. For options at or out of the money, both techniques provide accurate option values. In-the-money options have minor mispricing errors, but on a proportionate basis the errors are trivial. The overwhelming advantage of using the quadratic approximation, however, lies in the fact that its computational cost is approximately 100 times less than the compound-option approximation.

C.2. *Commodity Futures Option Values*

In Table III, the simulation results for futures options are reported. With the cost-of-carry parameter set equal to 0, the quadratic approximation shows even more precision across the parameter ranges considered. The largest errors are on the order of one tenth of one percent.

C.3. *Stock Option Values*

Table IV contains the simulation results for the special case where the cost of carry is equal to the riskless rate of interest, that is, for American options written

[10] Recall that, when the cost of carry is greater than or equal to the riskless rate of interest, the American call option will not be exercised early.

[11] Here, the finite-difference method is assumed to provide the "true" value of the American commodity option.

Table II

Theoretical American Commodity Option Values Using Finite-Difference, Compound-Option, and Quadratic Approximation Methods (Cost of Carry (b) = 0.04 and Exercise Price (X) = 100)

Option Parameters[a]	Commodity Price S	Call Options: European $c(S,T)$[b]	Call Options: American $C(S,T)$ Finite-Difference Method[c]	Call Options: American $C(S,T)$ Compound-Option Method[d]	Call Options: American $C(S,T)$ Quadratic Approximation Method[e]	Put Options: European $p(S,T)$[b]	Put Options: American $P(S,T)$ Finite-Difference Method[c]	Put Options: American $P(S,T)$ Compound-Option Method[d]	Put Options: American $P(S,T)$ Quadratic Approximation Method[e]
	80	0.05	0.05	0.05	0.05	18.87	20.00	19.99	20.00
$r = 0.08$, $\sigma = 0.20$, $T = 0.25$	90	0.85	0.85	0.85	0.85	9.76	10.22	10.25	10.18
	100	4.44	4.44	4.44	4.44	3.46	3.55	3.54	3.54
	110	11.66	11.66	11.66	11.66	0.78	0.79	0.79	0.80
	120	20.90	20.90	20.90	20.90	0.11	0.11	0.11	0.12
	80	0.05	0.05	0.05	0.05	18.68	20.00	19.99	20.00
$r = 0.12$, $\sigma = 0.20$, $T = 0.25$	90	0.84	0.84	0.84	0.84	9.67	10.20	10.23	10.16
	100	4.40	4.40	4.40	4.40	3.42	3.52	3.52	3.53
	110	11.55	11.55	11.55	11.55	0.77	0.78	0.78	0.79
	120	20.69	20.69	20.69	20.69	0.11	0.11	0.11	0.12
	80	1.29	1.29	1.29	1.29	20.11	20.59	20.60	20.53
$r = 0.08$, $\sigma = 0.40$, $T = 0.25$	90	3.82	3.82	3.82	3.82	12.74	12.95	12.94	12.93
	100	8.35	8.35	8.35	8.35	7.36	7.46	7.46	7.46
	110	14.80	14.79	14.80	14.80	3.91	3.95	3.95	3.96
	120	22.71	22.71	22.71	22.72	1.93	1.94	1.94	1.95
	80	0.41	0.41	0.41	0.41	18.08	20.00	19.96	20.00
$r = 0.08$, $\sigma = 0.20$, $T = 0.50$	90	2.18	2.18	2.18	2.18	10.04	10.75	10.79	10.71
	100	6.50	6.50	6.50	6.50	4.55	4.77	4.75	4.77
	110	13.42	13.42	13.42	13.42	1.68	1.74	1.74	1.76
	120	22.06	22.06	22.06	22.06	0.51	0.53	0.53	0.55

[a] The notation in this column is as follows: r = riskless rate of interest; σ = standard deviation of the commodity price-change relative; and T = time to expiration.

[b] Values are computed using equations (5) and (6).

[c] Values are computed using the implicit finite-difference method with commodity price steps of 0.10 and time steps of 0.20 days or 0.0005479 years.

[d] Values are computed using the three-point extrapolation of the compound-option valuation approach.

[e] Values are computed using the quadratic approximation equations (20) and (25).

Table III

Theoretical American Futures Option Values Using Finite-Difference, Compound-Option, and Quadratic Approximation Methods (Cost of Carry (b) = 0.00 and Exercise Price (X) = 100)

		Call Options				Put Options			
			American $C(F,T)$				American $P(F,T)$		
Option Parameters[a]	Futures Price F	European $c(F,T)$[b]	Finite-difference Method[c]	Compound-Option Method[d]	Quadratic Approximation Method[d]	European $p(F,T)$[b]	Finite-Difference Method[c]	Compound-Option Method[d]	Quadratic Approximation Method[e]
	80	0.04	0.04	0.04	0.04	19.64	20.00	20.00	20.00
$r = 0.08$, $\sigma = 0.20$, $T = 0.25$	90	0.70	0.70	0.70	0.70	10.50	10.59	10.58	10.58
	100	3.91	3.92	3.93	3.93	3.91	3.92	3.93	3.93
	110	10.74	10.82	10.81	10.81	0.94	0.94	0.94	0.94
	120	19.75	20.03	20.04	20.02	0.14	0.14	0.14	0.15
	80	0.04	0.04	0.04	0.04	19.45	20.00	19.99	20.00
$r = 0.12$, $\sigma = 0.20$, $T = 0.25$	90	0.69	0.69	0.69	0.70	10.40	10.53	10.53	10.53
	100	3.87	3.89	3.90	3.90	3.87	3.89	3.90	3.90
	110	10.63	10.76	10.76	10.75	0.94	0.93	0.93	0.93
	120	19.55	20.01	20.02	20.00	0.14	0.14	0.14	0.15
	80	1.16	1.16	1.16	1.17	20.77	20.94	20.94	20.93
$r = 0.08$, $\sigma = 0.40$, $T = 0.25$	90	3.52	3.53	3.53	3.53	13.32	13.39	13.39	13.39
	100	7.81	7.83	7.84	7.84	7.81	7.83	7.84	7.84
	110	14.01	14.08	14.08	14.08	4.21	4.22	4.22	4.23
	120	21.71	21.87	21.86	21.86	2.10	2.11	2.11	2.12
	80	0.30	0.30	0.30	0.30	19.51	20.06	20.09	20.04
$r = 0.08$, $\sigma = 0.20$, $T = 0.50$	90	1.70	1.71	1.71	1.72	11.31	11.48	11.47	11.48
	100	5.42	5.46	5.47	5.48	5.42	5.46	5.57	5.48
	110	11.73	11.90	11.89	11.90	2.12	2.14	2.14	2.15
	120	19.91	20.36	20.37	20.34	0.69	0.69	0.69	0.70

[a] The notation in this column is as follows: r = riskless rate of interest; σ = standard deviation of the commodity futures price-change relative; and T = time to expiration.

[b] Values are computed using equations (5) and (6).

[c] Values are computed using the implicit finite-difference method with commodity futures price steps of 0.10 and time steps of 0.20 days or 0.0005479 years.

[d] Values are computed using the three-point extrapolation of the compound-option valuation approach.

[e] Values are computed using the quadratic approximation equations (20) and (25).

Table IV

Theoretical American Put Option on Stock Values Using Finite-Difference, Compound-Option, Quadratic, and Johnson Approximation Methods (Exercise Price (X) = 100)

		Put Options				
			American $P(S,T)$			
Option Parameters[a]	Stock Price S	European $c(S,T)$[b]	Finite-Difference Method[c]	Compound-Option Method[d]	Quadratic Approximation Method[e]	Johnson Method[f]
	80	18.09	20.00	20.00	20.00	20.00
$b = r = 0.08$, $\sigma = 0.20$, $T = 0.25$	90	9.05	10.04	10.07	10.01	10.56
	100	3.04	3.22	3.21	3.22	3.21
	110	0.64	0.66	0.66	0.68	0.65
	120	0.09	0.09	0.09	0.10	0.09
	80	17.13	20.00	20.01	20.00	20.00
$b = r = 0.12$, $\sigma = 0.20$, $T = 0.25$	90	8.26	10.00	0.96	10.00	10.00
	100	2.63	2.92	2.91	2.93	2.90
	110	0.52	0.55	0.55	0.58	0.53
	120	0.07	0.07	0.07	0.08	0.07
	80	19.45	20.32	20.37	20.25	20.08
$b = r = 0.08$, $\sigma = 0.40$, $T = 0.25$	90	12.17	12.56	12.55	12.51	12.52
	100	6.94	7.11	7.10	7.10	7.12
	110	3.63	3.70	3.70	3.71	3.72
	120	1.76	1.79	1.79	1.81	1.80
	80	16.65	20.00	19.94	20.00	20.00
$b = r = 0.08$, $\sigma = 0.20$, $T = 0.50$	90	8.83	10.29	10.37	10.23	10.73
	100	3.79	4.19	4.17	4.19	4.17
	110	1.31	1.41	1.41	1.45	1.38
	120	0.38	0.40	0.40	0.42	0.39

[a] The notation in this column is as follows: b = cost of carrying underlying stock; r = riskless rate of interest; σ = standard deviation of the stock price-change relative; and T = time to expiration.

[b] Values are computed using equations (5) and (6).

[c] Values are computed using the implicit finite-difference method with stock price steps of 0.10 and time steps of 0.20 days or 0.0005479 years.

[d] Values are computed using the three-point extrapolation of the compound-option valuation approach.

[e] Values are computed using the quadratic approximation equations (20) and (25).

[f] Values are computed using the Johnson [12] method.

on non-dividend-paying stocks. Since the call will not rationally be exercised early, only put option values appear in the table. The quadratic approximation method used here is that of MacMillan [13].

With respect to the quadratic approximation and the compound-option approach, the results are qualitatively similar to the previous tables. Slightly larger mispricing errors occur for in-the-money options, but, even in the case where the volatility parameter is set equal to 0.40, the degree of mispricing is less than four tenths of one percent.

Unlike the previous tables, Table IV contains an additional column of values under the heading "Johnson Method." Johnson [12] provides a heuristic technique for valuing American put options on non-dividend-paying stocks.[12] His technique is slightly faster than the quadratic approximation; however, its validity appears to break down for put options that are slightly in the money. Consider the put option values for the first set of parameters. When the stock price is 80, all techniques yield an option value equal to 20.00. This is because the current stock price is below the critical stock price, so that the value of the American put is simply equal to its exercisable proceeds. However, if the stock price is 90, as seen in the second row of the table, the current stock price is in excess of the critical stock price and the approximation methods are invoked. While the quadratic approximation produces an absolute mispricing error of 0.03 (or 0.3 percent) relative to the finite-difference value, the Johnson technique produces a 0.52 (or 5.18 percent) error.

For the second and third set of option-pricing parameters, the Johnson technique produces reasonable values, but, for the fourth set of parameters, the first in-the-money put option again has a large mispricing error. This is indicative of the problems one faces when using heuristic procedures. While the option prices may be well behaved in general, they may lead to serious mispricing errors for arbitrary combinations of parameters.

C.4. Long-Term Option Values

The parameters of the options in Tables I through IV were chosen so as to represent typical exchange-traded options with times to expiration of less than six months. The most actively traded options, in fact, have maturities of less than three months. In the interest of completeness, however, it is worthwhile to point out that over-the-counter markets for long-term options are slowly developing, particularly in the area of U.S. Treasury obligations, and the impact of the time-to-expiration parameter on the accuracy of the approximation methods is of particular importance. For this reason, simulations are performed using times to expiration of up to three years. Table V contains the three-year time-to-expiration results.

The results in Table V show that all the approximation method results are weakened considerably. In some cases, the three-point extrapolation compound-

[12] When the underlying stock pays known discrete dividends, both the American call and put options written on the stock may be optimally exercised early. Roll [17] and Whaley [20] provide the analytic solution to the American call option-pricing problem where the stock pays known discrete dividends. Analytic solutions to the American put option-pricing problem have not been found; however, Geske and Johnson [10] and Blomeyer [4] provide heuristic techniques for approximating the put option values.

Table V

Theoretical American Commodity Option Values Using Finite-Difference, Compound-Option, Quadratic, and Johnson Approximation Methods (Exercise Price (X) = 100)

		Call Options				Put Options				
			American $C(S,T)$				American $P(S,T)$			
Option Parameters[a]	Commodity Price S	European $c(S,T)$[b]	Finite-Difference Method[c]	Compound-Option Method[d]	Quadratic Approximation Method[e]	European $p(S,T)$[b]	Finite-Difference Method[c]	Compound-Option Method[d]	Quadratic Approximation Method[e]	Johnson Method[f]
$b = -0.04$, $r = 0.08$, $\sigma = 0.20$,	80	1.93	2.34	2.31	2.52	24.78	25.66	25.59	26.25	
$T = 3.00$	90	3.75	4.76	4.71	4.97	19.62	20.08	20.05	20.64	
	100	6.36	8.49	8.54	8.67	15.25	15.50	15.51	15.99	
	110	9.75	13.79	14.08	13.88	11.67	11.80	11.83	12.22	
	120	13.87	20.89	21.29	20.88	8.81	8.88	8.91	9.23	
$b = 0.00$, $r = 0.08$, $\sigma = 0.20$,	80	3.79	3.98	3.99	4.20	19.52	22.20	22.35	22.40	
$T = 3.00$	90	6.81	7.25	7.23	7.54	14.68	16.21	16.18	16.50	
	100	10.82	11.70	11.65	12.03	10.82	11.70	11.65	12.03	
	110	15.71	17.31	17.28	17.64	7.85	8.37	8.34	8.69	
	120	21.35	24.02	24.11	24.30	5.62	5.93	5.93	6.22	
$b = 0.04$, $r = 0.08$, $\sigma = 0.20$,	80	6.88	6.88	6.88	6.97	14.59	20.35	20.60	20.33	
$T = 3.00$	90	11.49	11.48	11.49	11.62	10.33	13.50	13.69	13.56	
	100	17.20	17.19	17.22	17.40	7.17	8.94	8.95	9.11	
	110	23.80	23.80	23.85	24.09	4.90	5.91	5.85	6.12	
	120	31.08	31.08	31.18	31.49	3.32	3.90	3.85	4.12	
$b = 0.08$, $r = 0.08$, $\sigma = 0.20$,	80					10.25	20.00	19.44	20.00	20.00
$T = 3.00$	90					6.78	11.69	11.96	11.63	12.89
	100					4.41	6.93	7.06	6.96	6.69
	110					2.83	4.15	4.13	4.26	3.71
	120					1.80	2.51	2.45	2.64	2.16

[a] The notation in this column is as follows: b = cost of carrying underlying stock; r = riskless rate of interest; σ = standard deviation of the commodity price-change relative; and T = time to expiration.

[b] Values are computed using equations (5) and (6).

[c] Values are computed using the implicit finite-difference method with commodity price steps of 0.10 and time steps of 0.20 days or 0.0005479 years.

[d] Values are computed using the three-point extrapolation of the compound-option valuation approach.

[e] Values are computed using the quadratic approximation equations (20) and (25).

[f] Values are computed using the Johnson [12] method.

option method does better than the quadratic approximation, and in other cases vice versa. The Johnson technique produces the largest mispricing errors for the American put option on a non-dividend-paying stock.

Based on the results of Table V and the other simulation results (not reported here) using time-to-expiration parameters of between 0.5 and 3 years, it is reasonable to use either the three-point compound-option extrapolation method or the quadratic approximation method for pricing commodity options with less than one year to expiration, with the obvious preference being for the quadratic approximation method because of its computational expediency. For times to expiration beyond one year, finite-difference or binomial option-pricing methods should be used to ensure pricing accuracy.

III. Summary

More than thirty commodity option and commodity futures option contracts now trade in a variety of markets in the U.S. and Canada. These options are, in general, American style and, as such, are exercisable at any time up to and including the expiration day of the option. Previous attempts at pricing these options have been accurate but computationally expensive. This paper provides simple, inexpensive approximations for valuing exchange-traded American call and put options written on commodities as well as commodity futures contracts.

Appendix

Derivation of Analytic Approximation of Critical Commodity Price S^*: Equation (19) shows that the critical commodity price is an increasing function of time to expiration of the option bounded by the exercise price when $T = 0$ and by

$$S^*(\infty) = X/[1 - 1/q_2(\infty)], \tag{A1}$$

where $q_2(\infty) = [-(N-1) + \sqrt{(N-1)^2 + 4M}]/2$ when $T = +\infty$. To derive an approximate analytic equation for the critical commodity price as a function of time, consider the call option holder's dilemma when the time to expiration of the option is some arbitrarily small time increment, Δ. If the call is exercised at time Δ, the exercisable proceeds are $S(\Delta) - X$, which will earn interest to become $[S(\Delta) - X](1 + r\Delta)$ at $T = 0$. On the other hand, if the option holder chooses to leave the call position open, the worth of the call is equal to the expected terminal value of the option, $\mathrm{E}[S(0) - X \mid S(0) > X]$. Thus, the critical commodity price above which the call option holder will choose to exercise early is determined by

$$[S^*(\Delta) - X](1 + r\Delta) = \mathrm{E}[S^*(0) - X \mid S^*(0) > X]. \tag{A2}$$

To evaluate the right-hand side of (A2), represent the commodity price at the expiration of the option using the Cox-Ross-Rubinstein [7] risk-neutral binomial approach, that is,

$$S^*(0) = S^*(\Delta)(1 + b\Delta \pm \sigma\sqrt{\Delta}). \tag{A3}$$

Equation (A3) says that, in a risk-neutral world, the expected rate of return on commodity S is equal to its cost of carry, $b\Delta$, plus or minus the stochastic component, $\sigma\sqrt{\Delta}$, with equal probabilities. Thus, the expected value of holding the call to expiration is

$$\mathrm{E}[S^*(0) - X \mid S^*(0) > X] = 0.5[S^*(1 + b\Delta + \sigma\sqrt{\Delta}) - X], \tag{A4}$$

and, if (A4) is substituted into (A2), the critical commodity price is determined by

$$(S^*(\Delta) - X)(1 + r\Delta) = 0.5[S^*(1 + b\Delta + \sigma\sqrt{\Delta}) - X]. \tag{A5}$$

Rearranging equation (A5) to isolate $S^*(\Delta)$ provides

$$S^*(\Delta) = [X(1 + 2\Delta r)]/[1 + (2r - b)\Delta - \sigma\sqrt{\Delta}], \tag{A6}$$

which, in turn, provides the approximations

$$\begin{aligned} S^*(\Delta) &\approx X(1 + 2\Delta r)[1 - (2r - b)\Delta + \sigma\sqrt{\Delta}] \\ &\approx X(1 + b\Delta + \sigma\sqrt{\Delta}). \end{aligned} \tag{A7}$$

Equation (A7) ignores terms of order higher than Δ. Moreover, Δ is assumed to be small enough to make opportunities of exercising the call at intermediate times before expiration negligible. Therefore, equation (A7) holds exactly only in the case where Δ approaches 0.

To approximate S^* for arbitrary times to expiration, expand $S^*(0)$ around $S^*(\Delta)$, that is,

$$S^*(0) = S^*(\Delta) + (\delta S^*/\delta T)_{T=\Delta}\Delta. \tag{A8}$$

(The reason for choosing $T = \Delta$ in lieu of $T = 0$ as the origin of the expression (A8) is that at $T = 0$ the slope is discontinuous.) Substituting (A7) for $S^*(\Delta)$ and recalling that $S^*(0) = X$, it follows that the critical commodity price satisfies the differential equation

$$\delta S^*/\delta T = S^*(0)(b + \sigma/\sqrt{\Delta}) \tag{A9}$$

in a neighborhood of $T = 0$, with boundary condition $S^*(0) = X$. The general solution of (A9) is of the exponential form, with exponent $(bT + 2\sigma\sqrt{T})$.

Now, drawing the results together, the critical commodity price function is bounded at $T = 0$ and at $T = +\infty$ and has a slope described by the differential equation (A9). An appropriate final form for the critical commodity price of an American call option is therefore

$$S^* = X + [S^*(\infty) - X][1 - e^{h_2}], \tag{A10}$$

where $h_2 = -(bT + 2\sigma\sqrt{T})\{X/[S^*(\infty) - X]\}$. A parallel analysis can be made in deriving an approximate analytical critical commodity price equation for the American put option.

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